

# LINEAR AND PROJECTIVE BOUNDARY OF NILPOTENT GROUPS

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**ABSTRACT.** We introduce and study a general concept of boundaries of metric spaces which is based on the principle that sequences tend to the same point at infinity whenever they stay sublinearly close to each other with respect to some reference point. Such a boundary is determined by an arbitrarily chosen family of unbounded subsets. Examples are the boundary of a CAT(0) space, for which the chosen sets are geodesic rays, and the concept of so-called bundles in infinite graphs due to Bonnington, Richter and Watkins, where the chosen sets are one-way infinite paths.

Our particular interest lies in those boundaries which we get by choosing as families of unbounded subsets the positive or arbitrary powers, respectively, of non-torsion group elements in a finitely generated group (or more general by choosing positive or arbitrary powers of non-compact group elements in compactly generated, locally compact Hausdorff groups). We determine these boundaries for nilpotent groups as a disjoint union of spheres or as disjoint union of projective spaces, respectively. The number of disjoint spheres or projective spaces is equal to the nilpotency class. The dimensions of the spheres or projective spaces are each one less than the torsion-free rank (or compact-free dimension) of the corresponding commutative quotient obtained from the descending central series.

In addition we apply this concept to locally finite graphs with polynomial growth and random walks on nilpotent groups.

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## 1. INTRODUCTION

The theories of asymptotic properties of finitely generated groups and of locally finite graphs join a long history of mutual influence. In the early 1940s Freudenthal [Fre42] and Hopf [Hop44] introduced the notion of ends of metric spaces and groups and in 1964 Halin [Hal64] introduced/rediscovered the notion of ends of graphs. Starting from these concepts, both theories developed refinements:

Culminating in Gromov’s seminal work [Gro93] geometric group theory started to consider groups as geometric objects and asked for more sophisticated asymptotic invariants to describe the large-scale geometry of groups. One of the most prominent examples of such an invariant is the Gromov boundary for word-hyperbolic groups which was later on generalized to CAT(0) groups and Bruhat-Tits buildings for simple algebraic groups.

In graph theory in the 1990s Jung [Jun93] and Jung, Niemayer [JN95] introduced a refinement of ends of graphs called b-fibers and d-fibers, respectively. The basic idea behind fibers is to consider points at infinity as equivalence classes of rays (infinite paths) which have bounded Hausdorff distance. In 2005 Bonnington, Richter and Watkins [BRW07] modified this concept by considering rays as equivalent whenever they stay sublinearly close to each other with respect to a reference point. Moreover, they generalized the property of staying sublinearly close to the concept of staying at a distance which grows slower than some given increasing function. They were able to use this concept to prove some nice results on infinite planar graphs, but the boundary, whose elements have been called “bundles”, was not topologized and not considered for groups or vertex-transitive graphs.

It is a natural question up to which extent the idea of the bundles can be applied to group theory to obtain an intrinsic notion of boundary for arbitrary finitely generated groups (or more general for compactly generated, locally compact Hausdorff groups). Such groups carry word metrics which are defined in terms of a finite (or compact) generating set, and boundaries may be defined with respect to a chosen word metric. As we are interested in groups as such, boundaries of them should be independent from the choice of the word metric. Thus we are looking for a boundary which is invariant under the change of the generating set. Moreover, it is nearby to ask for the boundary to be a geometric property, i.e. invariant under quasi-isomorphisms.

The boundary of CAT(0) space is constructed by considering equivalence classes of non-compact geodesic rays, see [BH99]. If, however, the group in question is not a CAT(0) space, the set of non-compact geodesic rays might be empty, or its structure might be hard to understand, or there might be just too many geodesic rays at all. For example consider the group  $\mathbb{Z}^n$  equipped with the usual  $\ell^1$ -metric. Then any ray that moves along edges to the right and upwards is geodesic, whence the set of geodesics is huge. Of course we could solve that problem by first considering the boundary of Lie groups and then translate it back to lattices, but this would only work for Lie groups and their lattices and leaves us with the problem of understanding geodesics in Lie groups. Instead of geodesics we consider cyclic subgroups and cyclic subsemigroups, i.e. sets of the form  $\{g^n : n \in \mathbb{Z}\}$  and  $\{g^k : k \in \mathbb{N}_0\}$  for  $g \in G$ , respectively, and use a word metric on the group to compare these sets. If we start the boundary construction with all unbounded cyclic subsemigroups, we get a sphere as boundary of  $\mathbb{R}^d$  and  $\mathbb{Z}^d$ . And this is what we were looking for.

A general framework is constructed in Section 2 by defining a pseudometric on the set of unbounded subsets of a metric space similar to Tit's angular metric. If  $(X, d)$  is a metric space with reference point  $o \in X$  and  $R, S$  are unbounded subsets, their distance  $t(R, S)$  is defined to be square root of the infimum of  $\alpha \geq 0$ , such that

$$S \subseteq \bigcup_{x \in R} B(x, \alpha d(o, x) + a) \quad \text{and} \quad R \subseteq \bigcup_{y \in S} B(y, \alpha d(o, y) + a)$$

for some  $a \geq 0$ , where  $B(x, r)$  is the closed ball with center  $x$  and radius  $r$  in  $(X, d)$ . Then  $t$  is a pseudometric on the unbounded subsets of  $(X, d)$  and the Kolmogorov quotient is a complete metric space, see Proposition 2.6 and Theorem 2.9. We call  $t$  the *angle metric of unbounded subsets*. Given some family  $\mathcal{E}$  of unbounded subsets the associated “boundary” of  $X$  is the closure of all equivalence classes which contains an element from  $\mathcal{E}$  in the Kolmogorov quotient.

This metric is preserved up to bi-Lipschitz-equivalence under quasi-isometries of the underlying space as we show in Section 3 (see Theorem 3.3).

In Section 4 we study the connection of our general construction to the boundary at infinity of complete CAT(0) spaces. Proposition 4.2 shows that the boundary at infinity of complete CAT(0) spaces equipped with the angular metric can be obtained by the boundary construction outlined above using the set of unbounded geodesics up to bi-Hölder equivalence.

In Section 5 we apply this concept to compactly generated, locally compact, Hausdorff groups. As mentioned above good candidates for unbounded sets in a group  $G$  turn out to be the family of all unbounded cyclic subgroups and the family of all unbounded cyclic subsemigroups. The former ones yield what we call the *projective boundary*, the latter ones the *linear boundary* of  $G$ . As requested earlier, the linear boundary of  $\mathbb{Z}^d$  turns out to be the  $(d - 1)$ -dimensional sphere, its projective boundary is the  $(d - 1)$ -dimensional projective space.

In Section 6 we identify the linear and projective boundary of connected, nilpotent Lie groups and of finitely generated, nilpotent groups. Assume that  $G$  is either of these and

$$G = \gamma_1(G) \supseteq \gamma_2(G) \supseteq \cdots \supseteq \gamma_c(G) \not\supseteq \gamma_{c+1}(G) = 1$$

is the descending central series of  $G$ . Let  $\nu(i)$  denote the compact-free dimension or the torsion-free rank of the commutative group  $\gamma_i(G)/\gamma_{i+1}(G)$ . Then Theorem 6.1 states that the linear boundary  $\mathcal{L}G$  is homeomorphic to the disjoint union of  $c$  spheres:

$$\mathcal{L}G = \mathbb{S}^{\nu(1)-1} \uplus \mathbb{S}^{\nu(2)-1} \uplus \cdots \uplus \mathbb{S}^{\nu(c)-1}.$$

Analogously, the projective boundary  $\mathcal{P}G$  is homeomorphic to the disjoint union of projective spaces:

$$\mathcal{P}G = \mathbb{P}^{\nu(1)-1} \uplus \mathbb{P}^{\nu(2)-1} \uplus \cdots \uplus \mathbb{P}^{\nu(c)-1}.$$

Here  $\mathbb{S}^d$  is the  $d$ -dimensional sphere and  $\mathbb{P}^d$  is the  $d$ -dimensional projective space.

Applying the description of automorphism groups of connected, vertex-transitive graphs of polynomial growth which is supplied by Trofimov's well known paper [Tro84], we extend this theorem to vertex-transitive graphs with polynomial growth in Section 7 and obtain a description of their linear and projective boundaries, see Theorem 7.9 and Corollary 7.12. It is even possible to give a simple assumption under which the linear (projective) boundary of a vertex-transitive graph  $X$  with polynomial growth and the linear (projective) boundary of its automorphism group  $\text{Aut } X$  are bi-Lipschitz-equivalent.

In Section 8 it is shown that assuming a certain geometric condition on the group, the linear (or projective) boundary can be attached to the group, i.e. there is a topology on the disjoint union of group and boundary, such that the subspace topology on the group is the group topology and the subspace topology on the boundary is induced by the angle metric of unbounded subsets, see Lemma 8.1 and Proposition 8.3. This topology is reminiscent of the cone topology in the setting of CAT(0) spaces. This certain geometric condition holds for connected, nilpotent Lie groups or for finitely generated, nilpotent groups, see Lemma 8.4.

In Section 9 we finally show that the linear boundary of connected, nilpotent Lie groups (or finitely generated, nilpotent groups) is naturally linked to the asymptotic behavior of a generic trajectory of a random walk with “drift” on the group. It is shown that there is an equivalence class in the linear boundary, such that a generic trajectory of the walk considered as an unbounded subset lies in that class. Hence, if the linear boundary is attached to the group, almost surely the random walk converges to a point in the boundary, see Theorem 9.1.

Finally, we provide some details on compactly generated, locally compact Hausdorff groups and on word metrics on such groups in Appendix A and necessary technical tools for nilpotent groups in Appendix B, which are used to prove our result on the linear (and projective) boundary for connected, nilpotent Lie groups and their discrete counterpart.

## 2. GENERAL CONSTRUCTION

Let  $(X, d)$  be a metric space. We write  $\mathcal{U}$  to denote the family of unbounded subsets of  $(X, d)$ . The closed and open ball with center  $x \in X$  and radius  $r \geq 0$  in  $(X, d)$  are denoted by

$$B(x, r) = \{y \in X : d(y, x) \leq r\} \quad \text{and} \quad U(x, r) = \{y \in X : d(y, x) < r\},$$

respectively. Let  $o$  be a fixed reference point, let  $R \subseteq X$ , and let  $\alpha$  and  $a$  be nonnegative real numbers. We set

$$\alpha R + a = \bigcup_{x \in R} B(x, \alpha d(o, x) + a)$$

and write  $\alpha R$  instead of  $\alpha R + 0$ .

*Remark.* The notation  $\alpha R + a$ , apart from being unusual, turns out to be convenient for computations involving sets of this form. Mostly this notation will be used if  $X$  is a metric space, so no confusion should occur. However, if  $X$  is a linear space too,  $\alpha R + a$  will always be used in the above meaning and never means a linearly scaled and translated set. Furthermore, it should be stressed that  $0R = R$  and

$$0R + a = \bigcup_{x \in R} B(x, a),$$

which is often called  $a$ -expansion of  $R$  or generalized ball of radius  $a$  around  $R$ .

**Lemma 2.1.** *Let  $R \in \mathcal{U}$  and  $\alpha > 1$ . Then  $\alpha R = X$ .*

*Proof.* Let  $x$  be any point in  $X$ . Since  $R$  is unbounded, there is an element  $y \in R$  such that  $(\alpha - 1)d(o, y) \geq d(o, x)$ . Hence

$$d(x, y) \leq d(x, o) + d(o, y) \leq (\alpha - 1)d(o, y) + d(o, y) = \alpha d(o, y),$$

and  $x \in B(y, \alpha d(o, y)) \subseteq \alpha R$ . □

**Lemma 2.2.** *Let  $R, S$ , and  $T$  be subsets of  $X$ . If  $T \subseteq \beta S + b$  and  $S \subseteq \alpha R + a$  then  $T \subseteq (\alpha + \alpha\beta + \beta)R + \beta a + a + b$ .*

*Proof.* Let  $z$  be in  $T$ . Since the sets  $\alpha R + a$  and  $\beta S + b$  are defined as unions of balls,  $z$  is in  $B(y, \beta d(o, y) + b)$  for some  $y \in S$  and  $y$  is in  $B(x, \alpha d(o, x) + a)$  for some  $x \in R$ . Set  $c = d(o, x)$ . Then  $d(x, y) \leq \alpha c + a$ . By the triangle inequality,

$$d(o, y) \leq d(o, x) + d(x, y) \leq (\alpha + 1)c + a.$$

Hence

$$d(y, z) \leq \beta d(o, y) + b \leq (\alpha\beta + \beta)c + \beta a + b.$$

Finally,

$$d(x, z) \leq d(x, y) + d(y, z) \leq (\alpha + \alpha\beta + \beta)c + \beta a + a + b.$$

This means that  $z \in (\alpha + \alpha\beta + \beta)R + \beta a + a + b$ .  $\square$

**Definition 2.3.** For two subsets  $R, S \subseteq X$  let  $s^+(R, S)$  be the infimum of all  $\alpha \geq 0$  such that  $S \subseteq \alpha R + a$  for some  $a \geq 0$ . Set

$$s(R, S) = \max\{s^+(R, S), s^+(S, R)\}$$

and  $t(R, S) = \sqrt{s(R, S)}$ . If  $R, S \in \mathcal{U}$  and  $s(R, S) = 0$  then  $R$  and  $S$  are called *linearly equivalent* and we write  $R \sim S$ .

*Remark.* The functions  $s^+$ ,  $s$ ,  $t$  depend on the metric space  $(X, d)$ . In order to emphasize the underlying metric space  $(X, d)$  we write  $s_X^+$  or  $s_{(X, d)}^+$  and analogously for  $s$  and  $t$ . Similarly, we write  $\mathcal{U}_X$  or  $\mathcal{U}_{(X, d)}$  instead of  $\mathcal{U}$ , if it is necessary to specify the metric space.

**Lemma 2.4.** *Let  $R, S$  be two subsets of  $X$ . Then  $s^+(R, S)$  and therefore  $s(R, S)$ ,  $t(R, S)$  do not depend on the reference point  $o$  in  $X$ .*

*Proof.* Let  $o, p \in X$  and set  $c = d(o, p)$ . We write  $s_o^+(R, S)$  in order to emphasize the reference point  $o$ . Furthermore, write  $C_o(R, \alpha, a)$  to denote the set  $\alpha R + a$  with respect to the reference point  $o$ . For  $\alpha > s_o^+(R, S)$  there is a number  $a > 0$  such that  $S \subseteq C_o(R, \alpha, a)$ . Hence for  $y \in S$  we can find a point  $x \in R$  such that  $d(y, x) \leq \alpha d(o, x) + a$ . The triangle inequality implies that

$$d(y, x) \leq \alpha(d(p, x) + d(p, o)) + a = \alpha d(p, x) + \alpha c + a.$$

Therefore  $S \subseteq C_p(R, \alpha, \alpha c + a)$  and thus  $s_p^+(R, S) \leq s_o^+(R, S)$ . The reversed inequality is obtained by changing the rôle of  $o$  and  $p$ .  $\square$

**Lemma 2.5.** *Let  $R, S \in \mathcal{U}$ . Then  $s^+(R, S)$  is the infimum of all  $\alpha \geq 0$  such that  $S \setminus U(o, r) \subseteq \alpha R$  for some  $r \geq 0$ .*

*Proof.* We write  $\sigma^+(R, S)$  to denote the infimum of all  $\alpha \geq 0$  such that  $S \setminus U(o, r) \subseteq \alpha R$  for some  $r \geq 0$ . First we show that  $s^+(R, S) \leq \sigma^+(R, S)$ . Assume that  $\alpha > \sigma^+(R, S)$  and  $r \geq 0$  such that  $S \setminus U(o, r) \subseteq \alpha R$ . Set  $a = r + d(o, R)$ , where  $d(o, R) = \inf\{d(o, x) : x \in R\}$ . Then, by triangle inequality,  $S \subseteq \alpha R + a$ . Therefore  $s^+(R, S) \leq \alpha$  and hence  $s^+(R, S) \leq \sigma^+(R, S)$ .

Now we prove the reversed inequality: Let  $\alpha > s^+(R, S)$  and set  $\varepsilon = \frac{1}{2}(\alpha - s^+(R, S)) > 0$ . Then, by definition of  $s^+(R, S)$ , there exists a constant  $a \geq 0$  such that  $S \subseteq (\alpha - \varepsilon)R + a$ . We claim that  $S \setminus U(o, r) \subseteq \alpha R$  holds for  $r = \frac{a}{\varepsilon}(1 + \alpha)$ . Let

178  $y \in S$ . Then there is a  $x \in R$  with  $d(y, x) \leq (\alpha - \varepsilon)d(o, x) + a$ . Using the triangle  
179 inequality yields

$$d(o, y) \leq d(o, x) + d(y, x) \leq d(o, x) + (\alpha - \varepsilon)d(o, x) + a \leq (1 + \alpha - \varepsilon)d(o, x) + a,$$

180 which implies

$$d(o, x) \geq \frac{d(o, y) - a}{1 + \alpha - \varepsilon}.$$

181 If  $d(o, y) \geq r$  then we obtain

$$a \leq \varepsilon \cdot \frac{d(o, y) - a}{1 + \alpha - \varepsilon} \leq \varepsilon d(o, x)$$

182 and

$$d(y, x) \leq (\alpha - \varepsilon)d(o, x) + a \leq (\alpha - \varepsilon)d(o, x) + \varepsilon d(o, x) = \alpha d(o, y).$$

183 Therefore  $S \setminus U(o, r) \subseteq \alpha R$  and  $\sigma^+(R, S) \leq s^+(R, S)$ .  $\square$

184 **Proposition 2.6.** *The function  $s^+$  is a premetric on  $\mathcal{U}$  satisfying a weak form of  
185 the triangle inequality, i.e. if  $R, S, T$  are unbounded subsets of  $X$ , then*

- 186 •  $s^+(R, S) \in [0, 1]$  and  $s^+(R, R) = 0$ ,
- 187 •  $s^+(R, T) \leq s^+(R, S) + s^+(R, S)s^+(S, T) + s^+(S, T)$ .

188 Similarly,  $s$  is a symmetric premetric on  $\mathcal{U}$  satisfying the same weak triangle in-  
189 equality, i.e.

- 190 •  $s(R, S) \in [0, 1]$  and  $s(R, R) = 0$ ,
- 191 •  $s(R, S) = s(S, R)$ ,
- 192 •  $s(R, T) \leq s(R, S) + s(R, S)s(S, T) + s(S, T)$ .

193 Finally,  $t$  is a pseudometric on  $\mathcal{U}$ , i.e.

- 194 •  $t(R, S) \in [0, 1]$  and  $t(R, R) = 0$ ,
- 195 •  $t(R, S) = t(S, R)$ ,
- 196 •  $t(R, T) \leq t(R, S) + t(S, T)$ .

197 *Proof.* The statements for  $s^+$  and  $s$  follow from the definition and from the Lem-  
198 mas 2.1 and 2.2.

199 It remains to show that  $t$  satisfies the triangle inequality. Let  $R, S$  be unbounded  
200 subsets of  $X$ . Then

$$\begin{aligned} s(R, T) &\leq s(R, S) + s(R, S)s(S, T) + s(S, T) \\ &\leq s(R, S) + 2\sqrt{s(R, S)s(S, T)} + s(S, T) \end{aligned}$$

201 which implies

$$t(R, T) = \sqrt{s(R, T)} \leq \sqrt{s(R, S)} + \sqrt{s(S, T)} = t(R, S) + t(S, T). \quad \square$$

202 **Corollary 2.7.** *Assume that  $R, S, T$  are unbounded subsets of  $X$ . If  $s^+(S, T) = 0$   
203 then  $s^+(R, T) \leq s^+(R, S)$  and  $s^+(S, R) \leq s^+(T, R)$ . Therefore, if  $S \sim T$ , then  
204  $s^+(R, S) = s^+(R, T)$ ,  $s^+(S, R) = s^+(T, R)$ , and  $s(R, S) = s(R, T)$ ,  $t(R, S) =$   
205  $t(R, T)$ .*

206 *Proof.* Using Proposition 2.6 and  $s^+(S, T) = 0$  we get

$$s^+(R, T) \leq s^+(R, S) + s^+(R, S)s^+(S, T) + s^+(S, T) = s^+(R, S)$$

207 and

$$s^+(S, R) \leq s^+(S, T) + s^+(S, T)s^+(T, R) + s^+(T, R) = s^+(T, R).$$

208 The remaining claims follow, since  $S \sim T$  implies  $s^+(S, T) = s^+(T, S) = 0$ .  $\square$

**Corollary 2.8.** *Linear equivalence is an equivalence relation on unbounded subsets and the functions  $s^+$ ,  $s$ ,  $t$  are well-defined on the quotient space  $\mathcal{U}/\sim$ .*

*Proof.* Reflexivity and symmetry follow immediately from the definition. Transitivity follows from Corollary 2.7. Let  $R_1, R_2, S_1, S_2$  be unbounded subsets and suppose  $s(R_1, R_2) = s(S_1, S_2) = 0$ . Corollary 2.7 implies that  $s^+(R_1, S_1) = s^+(R_2, S_1) = s^+(R_2, S_2)$ , whence  $s^+$  and therefore  $s, t$  are well-defined on equivalence classes.  $\square$

**Theorem 2.9.**  *$(\mathcal{U}/\sim, t)$  is a complete metric space.*

*Proof.* By Proposition 2.6 and Corollary 2.8  $(\mathcal{U}/\sim, t)$  is a metric space. It remains to prove that it is also complete.

Let  $(\xi_n)_{n \geq 0}$  be a Cauchy sequence in  $\mathcal{U}/\sim$ . Without loss of generality we may assume that  $s(\xi_n, \xi_m) \leq 1/2$  for all  $n, m$ . Choose representatives  $R_n \in \xi_n$ . Then for any  $\varepsilon > 0$  there is an index  $N$  such that  $s(R_n, R_m) < \varepsilon$  for  $n, m \geq N$ . Therefore there exists a function  $\varepsilon^*: \mathbb{N} \rightarrow (0, 1/2]$  such that  $\varepsilon^*$  is decreasing,  $\varepsilon^*(n) \rightarrow 0$  as  $n \rightarrow \infty$ , and  $s(R_m, R_n) < \varepsilon^*(m)$  for  $m \leq n$ .

According to Lemma 2.5 there are  $r(m, n) \geq 0$ , for  $m \leq n$ , such that

$$R_n \setminus U(o, r(m, n)) \subseteq \varepsilon^*(m)R_m \quad \text{and} \quad R_m \setminus U(o, r(m, n)) \subseteq \varepsilon^*(m)R_n.$$

Hence there is an increasing function  $r^*: \mathbb{N} \rightarrow [0, \infty)$  such that  $r^*(n) \geq r(m, n)$  for  $m \leq n$ . Applying Lemma 2.5 to  $R_m \setminus U(o, r^*(n))$  and  $R_n \setminus U(o, r^*(n))$  for  $m \leq n$  implies that there are  $q(m, n) \geq 0$  such that

$$\begin{aligned} R_n \setminus U(o, q(m, n)) &\subseteq \varepsilon^*(m)(R_m \setminus U(o, r^*(n))), \\ R_m \setminus U(o, q(m, n)) &\subseteq \varepsilon^*(m)(R_n \setminus U(o, r^*(n))). \end{aligned}$$

Thus there is an increasing function  $q^*: \mathbb{N} \rightarrow [0, \infty)$  such that  $q^*(n) \rightarrow \infty$  as  $n \rightarrow \infty$  and  $q^*(n) \geq q(m, n)$  for  $m \leq n$ .

Let  $x \in R_m$  with  $q^*(n) \leq d(o, x) < q^*(n+1)$  for some  $n \geq m$ . Then there is a  $y \in R_n$  such that

$$d(o, y) \geq r^*(n) \quad \text{and} \quad d(x, y) \leq \varepsilon^*(m)d(o, y).$$

Using the triangle inequality and  $\varepsilon^*(m) \leq 1/2$  we get

$$d(o, y) \leq d(o, x) + d(x, y) \leq d(o, x) + \varepsilon^*(m)d(o, y) \leq d(o, x) + d(o, y)/2$$

and

$$(1) \quad d(o, y) \leq 2d(o, x) < 2q^*(n+1).$$

We write  $x^*$  to denote this element  $y$  and define the set  $S$  by

$$S = \bigcup_{m \geq 1} \{x^* : x \in R_m, d(o, x) \geq q^*(m)\}.$$

Then  $S$  is an unbounded subset of  $X$ . Note that if  $x \in S$  and  $d(o, x) \geq 2q^*(m)$  for some  $m$  then  $x \in R_n$  for some  $n \geq m$  due to the estimate in (1). We claim that  $s(S, R_m) \leq \varepsilon^*(m)$  for  $m \geq 1$ .

• Let  $x$  be an element of  $R_m$  with  $d(o, x) \geq q^*(m)$ . Then, by construction of  $S$ , there is a  $y \in S$  with  $d(x, y) \leq \varepsilon^*(m)d(o, y)$ . This implies

$$R_m \setminus U(o, q^*(m)) \subseteq \varepsilon^*(m)S.$$

- Let  $x$  be an element of  $S$  with  $d(o, x) \geq 2q^*(m)$ . Then  $x \in R_n$  for some  $n \geq m$ . This implies the lower bound  $d(o, x) \geq r^*(n)$ . By definition of  $r^*$  there is a  $y \in R_m$  such that  $d(x, y) \leq \varepsilon^*(m)d(o, y)$ . Hence

$$S \setminus U(o, 2q^*(m)) \subseteq \varepsilon^*(m)R_m.$$

This implies the claim. Let  $\zeta$  be the equivalence class of  $S$ . Then

$$s(\xi_m, \zeta) = s(R_m, S) \leq \varepsilon^*(m)$$

for  $m \geq 1$ . Therefore  $\xi_m$  converges to  $\zeta$  proving Cauchy completeness.  $\square$

**Definition 2.10.** We call  $t$  *angle metric of unbounded sets* (see Example 2.11).

If  $\Xi$  is a subset of  $\mathcal{U}/\sim$ , we write  $\text{cl}(\Xi)$  to denote the closure of  $\Xi$  in the metric space  $(\mathcal{U}/\sim, t)$ . Let  $\mathcal{E} \subseteq \mathcal{U}$  be a family of unbounded subsets of  $(X, d)$ . Define  $\mathcal{E}/\sim$  to be the set of equivalence classes in  $\mathcal{U}/\sim$  which contain at least one element from  $\mathcal{E}$ , this is

$$\mathcal{E}/\sim = \{[R] : R \in \mathcal{E}\} \subseteq \mathcal{U}/\sim,$$

where  $[R]$  is the equivalence class of  $R$  with respect to linear equivalence  $\sim$ . Note that  $\text{cl}(\mathcal{E}/\sim)$  is a well-defined subset of  $\mathcal{U}/\sim$  which is closed and hence Cauchy complete. Thus up to isometry  $(\text{cl}(\mathcal{E}/\sim), t)$  is the Cauchy completion of  $(\mathcal{E}/\sim, t)$ .

*Remark.* The definition of the set  $\mathcal{E}/\sim$  depends on the underlying metric space  $(X, d)$ . However, no confusion should occur, since the underlying metric space will be clear from the context. Moreover, the above definition of  $\mathcal{E}/\sim$  is somewhat unusual, since  $\mathcal{E}/\sim \subseteq \mathcal{U}/\sim$ . The reason for this definition is that we will use topological notions of  $(\mathcal{U}/\sim, t)$  for the subset  $\mathcal{E}/\sim$ . Furthermore, note that, if  $\sim_{\mathcal{E}}$  denotes the restriction of  $\sim$  to the set  $\mathcal{E}$  then

$$\mathcal{E}/\sim \rightarrow \mathcal{E}/\sim_{\mathcal{E}}, \quad \zeta \mapsto \zeta \cap \mathcal{E}$$

is a canonical bijection.

**Example 2.11.** Consider  $\mathbb{R}^n$  equipped with the usual  $\ell^2$ -metric. For a nonzero vector  $x \in \mathbb{R}^n$  let  $L_x$  denote the line  $\{\lambda x : \lambda \in \mathbb{R}\}$  and  $H_x$  the half-line  $\{\lambda x : \lambda \geq 0\}$ . Set  $\mathcal{L} = \{L_x : x \in \mathbb{R}^n, x \neq 0\}$  and  $\mathcal{H} = \{H_x : x \in \mathbb{R}^n, x \neq 0\}$ . Then  $\text{cl}(\mathcal{L}/\sim)$  is the projective space  $\mathbb{P}^{n-1}$  and  $\text{cl}(\mathcal{H}/\sim)$  is the sphere  $\mathbb{S}^{n-1}$ . If  $x, y \in \mathbb{R}^n \setminus \{0\}$  then

$$s(L_x, L_y) = \sin(\angle(L_x, L_y)) \quad \text{and} \quad s(H_x, H_y) = \sin(\min\{\frac{1}{2}\pi, \angle(H_x, H_y)\})$$

where  $\angle(L_x, L_y)$  is the smaller angle between the lines  $L_x$  and  $L_y$  and  $\angle(H_x, H_y)$  is the angle between the half-lines  $H_x, H_y$ .

The following examples show that the function  $s$  is not always a metric and that geodesics do not always yield a nice structure.

**Example 2.12.** Consider the 2-dimensional space  $\mathbb{R}^2$  with  $\ell^1$ -metric  $d_1$ . Let  $x_1 = (1, 0)$ ,  $x_2 = (2, 1)$ , and  $x_3 = (1, 1)$ . Set  $L_i = \{\lambda x_i : \lambda \in \mathbb{R}\}$  for  $i \in \{1, 2, 3\}$ . Then

$$s(L_1, L_2) = \frac{1}{2}, \quad s(L_2, L_3) = \frac{1}{3}, \quad s(L_1, L_3) = 1,$$

and the triangle inequality is not satisfied.

**Example 2.13.** Consider the metric space  $(\mathbb{Z}^2, d_1)$ , where  $d_1$  is the  $\ell^1$ -metric. In this discrete setting a geodesic ray is an infinite sequence  $(x_0, x_1, \dots)$  in  $\mathbb{Z}^2$  such that  $d(x_i, x_j) = |i - j|$ . Let  $\mathcal{G}$  be the family of all geodesic rays emanating from the origin. Furthermore, let  $\mathcal{E}$  be the family of all sets  $\{nx : n \in \mathbb{N}_0\}$  for  $x \in \mathbb{Z}^2, x \neq 0$ .



274 Then the space  $\text{cl}(\mathcal{G}/\sim)$  contains much more elements than  $\text{cl}(\mathcal{E}/\sim)$ . To see this set  
 275  $x_{2n} = (2^n - 1, 2^n - 1)$  and  $x_{2n+1} = (2^{n+1} - 1, 2^n - 1)$  for  $n \in \mathbb{N}_0$ . Join  $x_m$  and  $x_{m+1}$ ,  
 276  $m \in \mathbb{N}_0$ , by a geodesic path and let  $R$  denote the ray consisting of the union of these  
 277 finite geodesic paths. Obviously  $R$  is a geodesic ray and there is some  $\varepsilon > 0$  such  
 278 that  $s(R, S) \geq \varepsilon$  for all  $S \in \mathcal{E}$ .

### 279 3. QUASI-ISOMETRIES

280 **Definition 3.1.** Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces and let  $q > 0$ . A function  
 281  $f: X \rightarrow Y$  is called a *q-quasi-isometry* if

$$q^{-1}d_X(x, x') - q \leq d_Y(f(x), f(x')) \leq qd_X(x, x') + q$$

282 for all  $x, x' \in X$  and such that every closed ball in  $Y$  with radius  $q$  contains an  
 283 element of  $f(X)$ . We say that two metrics  $d_1$  and  $d_2$  on  $X$  are *quasi-isometrically*  
 284 *equivalent*, if the identity is a quasi-isometry from  $(X, d_1)$  to  $(X, d_2)$ .

285 **Lemma 3.2.** Let  $f: X \rightarrow Y$  be a *q-quasi-isometry* of the metric spaces  $(X, d_X)$   
 286 and  $(Y, d_Y)$ . Let  $R, S$  be unbounded subsets of  $X$ . Then  $f(R), f(S)$  are unbounded  
 287 subsets of  $Y$  and

$$q^{-2}s_X^+(R, S) \leq s_Y^+(f(R), f(S)) \leq q^2s_X^+(R, S).$$

288 *Proof.* We fix reference points  $o$  and  $f(o)$  in  $X$  and  $Y$ , respectively. First of all note  
 289 that

$$q^{-1}d_X(x, x') - q \leq d_Y(f(x), f(x')) \leq qd_X(x, x') + q$$

290 implies

$$q^{-1}d_Y(f(x), f(x')) - 1 \leq d_X(x, x') \leq qd_Y(f(x), f(x')) + q^2$$

291 for all  $x, x' \in X$ .

292 Let  $\alpha > s_X^+(R, S)$ . Then there is a number  $a$  such that  $S \subseteq \alpha R + a$ . Hence  
 293 for  $x' \in S$  there is a point  $x \in R$  with  $d_X(x', x) \leq \alpha d_X(o, x) + a$ . Since  $f$  is a  
 294 quasi-isometry, we get  $d_X(o, x) \leq qd_Y(f(o), f(x)) + q^2$ . This implies

$$\begin{aligned} d_Y(f(x'), f(x)) &\leq qd_X(x', x) + q \leq q\alpha d_X(o, x) + qa + q \\ &\leq q^2\alpha d_Y(f(o), f(x)) + q^3\alpha + qa + q, \end{aligned}$$

295 proving that

$$f(S) \subseteq q^2\alpha f(R) + q^3\alpha + qa + q$$

296 holds. Thus  $s_Y^+(f(R), f(S)) \leq q^2s_X^+(R, S)$ .

297 If  $s_X^+(R, S) = 0$  then  $s_Y^+(f(R), f(S)) \geq q^{-2}s_X^+(R, S)$  trivially holds. Hence we  
 298 assume that  $s_X^+(R, S) > 0$ . Then, for  $\alpha < s_X(R, S)$ ,  $S \subseteq \alpha R + a$  fails to be true for  
 299 all  $a \geq 0$ . Hence for every  $a \geq 0$  there exists a point  $x' \in S$  which is not contained  
 300 in  $\alpha R + a$ . Thus  $d_X(x', x) > \alpha d(o, x) + a$  for all  $x \in R$ . This implies

$$\begin{aligned} d_Y(f(x'), f(x)) &\geq q^{-1}d_X(x', x) - q > q^{-1}\alpha d_X(o, x) + q^{-1}a - q \\ &\geq q^{-2}\alpha d_Y(f(o), f(x)) + q^{-1}(a - \alpha) - q. \end{aligned}$$

301 Thus  $f(x')$  is not contained in  $q^{-2}\alpha f(R) + q^{-1}(a - \alpha) - q$ . Since  $a \geq 0$  was arbitrary,  
 302 this means that  $s_Y^+(f(R), f(S)) \geq q^{-2}s_X^+(R, S)$ .  $\square$

**Theorem 3.3.** *Let  $f: X \rightarrow Y$  be a  $q$ -quasi-isometry of the metric spaces  $(X, d_X)$  and  $(Y, d_Y)$ . Then  $f$  induces to bijection  $f: \mathcal{U}_X/\sim \rightarrow \mathcal{U}_Y/\sim$  which is bi-Lipschitz continuous:*

$$q^{-1}t_X(\zeta, \xi) \leq t_Y(f(\zeta), f(\xi)) \leq qt_X(\zeta, \xi)$$

*for all  $\zeta, \xi \in \mathcal{U}_X/\sim$ . In particular, if  $\mathcal{E}$  is a family of unbounded subsets in  $X$ , then  $f(\mathcal{E}/\sim) = f(\mathcal{E})/\sim$  and  $f(\text{cl}(\mathcal{E}/\sim)) = \text{cl}(f(\mathcal{E})/\sim)$ .*

*Proof.* Of course  $f(\mathcal{U}_X)$  is a subset of  $\mathcal{U}_Y$ . By Lemma 3.2 the function  $f: \mathcal{U}_X/\sim \rightarrow \mathcal{U}_Y/\sim$  which maps the equivalence class of an unbounded  $R \subseteq X$  to the equivalence class of  $f(R)$  is well-defined, one-to-one, and satisfies

$$q^{-1}t_X(\zeta, \xi) \leq t_Y(f(\zeta), f(\xi)) \leq qt_X(\zeta, \xi)$$

for all  $\zeta, \xi \in \mathcal{U}_X/\sim$ . Thus it remains to show that  $f$  is also onto. Let  $S$  be an unbounded subset of  $Y$ . Since  $f$  is a  $q$ -quasi-isometry, the set  $R = f^{-1}((0S + q) \cap f(X))$  is an unbounded subset of  $X$  and  $f(R) = (0S + q) \cap f(X) \sim S$ .  $\square$

#### 4. BOUNDARY AT INFINITY AND ANGULAR METRIC IN A CAT(0) SPACE

Let us recall the definitions of the boundary at infinity and the angular metric in a CAT(0) space. For more details we refer to the book of Bridson and Haefliger [BH99], see especially Chapter II.8 and Chapter II.9 therein. A *geodesic ray* in a metric space  $(X, d)$  is a curve  $c: [0, \infty) \rightarrow X$  such that  $d(c(x), c(y)) = |x - y|$  for all  $x, y \geq 0$ . The *boundary at infinity*  $\partial X$  of  $X$  is defined to be the set of equivalence classes of geodesic rays, where geodesic rays  $c, c'$  are equivalent whenever they stay at bounded distance, that is, if there is a constant  $K$ , such that  $d(c(x), c'(x)) \leq K$  for all  $x \in [0, \infty)$ . In the sequel we assume that  $X$  is a complete CAT(0) space.

For each point  $p$  in  $X$  and  $\xi$  in  $\partial X$  there is precisely one geodesic ray belonging to  $\xi$  which emanates from  $p$ . Then  $\angle_p(\xi, \zeta)$  for  $\xi, \zeta \in \partial X$  is defined to be the angle at  $p$  between the uniquely determined rays in  $\xi$  and  $\zeta$  which emanate from  $p$ . The angle between  $\xi$  and  $\zeta$  is defined by

$$\angle(\xi, \zeta) = \sup\{\angle_p(\xi, \zeta) : p \in X\}.$$

This yields a metric on  $\partial X$  called *angular metric* and  $(\partial X, \angle)$  is a complete metric space. For our purposes the following description of the angular metric is useful. Fix a reference point  $o$  in  $X$ . If  $\xi \in \partial X$ , we write  $c_\xi$  for the uniquely determined geodesic ray in  $\xi$  which emanates from  $o$  and  $R_\xi$  for the image of  $c_\xi$  in  $X$ , i.e.  $R_\xi = c_\xi([0, \infty))$ . Then

$$2 \sin\left(\frac{1}{2}\angle(\xi, \zeta)\right) = \lim_{x \rightarrow \infty} \frac{1}{x} d(c_\xi(x), c_\zeta(x)).$$

**Lemma 4.1.** *Let  $\xi, \zeta$  be elements in  $\partial X$ . Then*

$$s(R_\xi, R_\zeta) \leq 2 \sin\left(\frac{1}{2}\angle(\xi, \zeta)\right) \leq 4s(R_\xi, R_\zeta).$$

*Proof.* Note that  $d(o, c_\xi(x)) = d(o, c_\zeta(x)) = x$  for all  $x \in [0, \infty)$ , since  $c_\xi(0) = c_\zeta(0) = o$ . Suppose that  $\alpha > 2 \sin(\frac{1}{2}\angle(\xi, \zeta))$ . Then there exists a constant  $a \geq 0$ , such that  $d(c_\xi(x), c_\zeta(x)) \leq \alpha x$  for all  $x \geq a$ . This implies that

$$R_\xi \subseteq \alpha R_\zeta + a \quad \text{and} \quad R_\zeta \subseteq \alpha R_\xi + a.$$

Therefore  $s(R_\xi, R_\zeta) \leq \alpha$  which yields the lower bound.

If  $s(R_\xi, R_\zeta) \geq \frac{1}{2}$  then the upper bound is trivially true. Hence assume that  $s(R_\xi, R_\zeta) \leq \frac{1}{2}$  and fix some  $\alpha$ , such that  $s(R_\xi, R_\zeta) < \alpha \leq \frac{1}{2}$ . By Lemma 2.5 there

is a constant  $r \geq 0$ , such that  $R_\zeta \setminus U(o, r) \subseteq \alpha R_\xi$ . Hence, for any  $x \geq r$ , there is a  $y = y(x) \geq 0$ , such that

$$d(c_\zeta(x), c_\xi(y)) \leq \alpha d(o, c_\xi(y)) = \alpha y.$$

Using the triangle inequality and the estimate above yields  $y \leq x + \alpha y$  and  $x \leq y + \alpha y$ . It follows that  $|y - x| \leq \alpha y$  and  $y \leq 2x$ , since  $\alpha \leq \frac{1}{2}$ . Collecting the pieces we get

$$\begin{aligned} d(c_\zeta(x), c_\xi(x)) &\leq d(c_\zeta(x), c_\xi(y)) + d(c_\xi(y), c_\xi(x)) \\ &\leq \alpha d(o, c_\xi(y)) + |y - x| \\ &= 2\alpha y \leq 4\alpha x. \end{aligned}$$

and thus

$$2 \sin\left(\frac{1}{2}\angle(\xi, \zeta)\right) = \lim_{x \rightarrow \infty} \frac{1}{x} d(c_\zeta(x), c_\xi(x)) \leq 4\alpha. \quad \square$$

As a consequence of the previous lemma we get that two geodesic rays  $c$  and  $c'$  stay at bounded distance if and only if the subsets  $c([0, \infty))$  and  $c'([0, \infty))$  are linearly equivalent. Write  $\mathcal{G}$  to denote the family of all subsets of the form  $c([0, \infty))$ , where  $c$  is some geodesic ray in  $X$ .

**Proposition 4.2.** *Let  $X$  be a complete CAT(0) space. Then*

$$\partial X \rightarrow \mathcal{G}/\sim, \quad \xi \mapsto [R_\xi],$$

where  $[R_\xi]$  is the equivalence class of  $R_\xi$  with respect to linear equivalence, is one-to-one, onto, and bi-Hölder continuous:

$$\frac{1}{\pi}\angle(\xi, \zeta) \leq (t([R_\xi], [R_\zeta]))^2 \leq \angle(\xi, \zeta)$$

for all  $\xi, \zeta \in \partial X$ . Furthermore,  $\mathcal{G}/\sim$  is a closed subset of  $(\mathcal{U}/\sim, t)$ , since  $(\partial X, \angle)$  is a complete metric space.

*Proof.* Since  $\angle(\xi, \zeta) \in [0, \pi]$  and  $\frac{2}{\pi}x \leq \sin(x) \leq x$  for all  $x \in [0, \frac{\pi}{2}]$ , Lemma 4.1 yields  $\frac{1}{\pi}\angle(\xi, \zeta) \leq s(R_\xi, R_\zeta) \leq \angle(\xi, \zeta)$   $\square$

## 5. BOUNDARIES OF GROUPS

Let  $G$  be a group and  $d$  be a metric on  $G$ . Fix the identity element  $1 \in G$  as reference point. From an algebraic point of view it is natural to consider the families of unbounded cyclic subgroups and unbounded cyclic subsemigroups of the group  $G$ . Hence define

$$\mathcal{C}G = \{\langle g \rangle : g \in G, \langle g \rangle \in \mathcal{U}\}$$

and

$$\mathcal{C}^+G = \{\langle g \rangle^+ : g \in G, \langle g \rangle^+ \in \mathcal{U}\},$$

where  $\langle g \rangle^+ = \{g^n : n \in \mathbb{N}_0\}$  is the semigroup generated by  $g \in G$ . Note that, if  $\langle g \rangle^+ \in \mathcal{C}^+G$ , then  $\langle g \rangle \in \mathcal{C}G$ .

**Definition 5.1.** We define the *projective boundary* of  $G$  by  $\mathcal{P}G = \text{cl}(\mathcal{C}G/\sim)$  and the *linear boundary* by  $\mathcal{L}G = \text{cl}(\mathcal{C}^+G/\sim)$

*Remark.* Both,  $\mathcal{P}G$  and  $\mathcal{L}G$ , depend on the metric  $d$ . If it is necessary to emphasize this dependence, we write  $\mathcal{P}(G, d)$  and  $\mathcal{L}(G, d)$ , respectively.

**Lemma 5.2.** *If  $g \in \mathcal{C}G$  and  $h \in \mathcal{C}^+G$  then  $\langle g^n \rangle \sim \langle g \rangle$  and  $\langle h^n \rangle^+ \sim \langle h \rangle^+$  for all  $n \in \mathbb{N}$ . Furthermore, if  $d$  is left-invariant or right-invariant, then  $\langle g \rangle^+ \in \mathcal{C}^+G$  if and only if  $\langle g \rangle \in \mathcal{C}G$ .*

There are two interesting sources for metrics on a group  $G$ . If  $G$  is finitely generated (or more generally compactly generated), it is natural to consider word metrics on  $G$ . If  $G$  is a connected Lie group, it is natural to consider left-invariant Riemannian metrics on  $G$ . In this case  $G$  is also compactly generated and Corollary A.7 implies that any left-invariant Riemannian metric is quasi-isometrically equivalent to any word metric on  $G$ . Hence for our purposes it is sufficient to study the setting of compactly generated groups in more detail.

A topological group is called *compactly generated*, if there is a compact generating set  $K \subseteq G$ . In this case  $S = K \cup K^{-1}$  is a compact, symmetric (i.e.  $S = S^{-1}$ ), generating set. Set  $S^0 = \{1\}$  and  $S^n = \{s_1 \cdots s_n : s_1, \dots, s_n \in S\}$  for  $n \geq 1$ . Note that  $S^n$  is compact and symmetric for all  $n \geq 0$  and

$$G = \bigcup_{n \geq 0} S^n.$$

The *word metric*  $d$  of  $G$  with respect to  $S$  is defined by  $d(g, h) = \inf\{n : g^{-1}h \in S^n\}$ . The metric  $d$  is left-invariant and induces the discrete topology on  $G$  which is in general different from the group topology. In the sequel we consider the class of compactly generated, locally compact Hausdorff groups. Some facts about such groups and their word metrics are provided by Appendix A. Finitely generated groups fit in this setting (in this case a finitely generated group is equipped with the discrete topology). If not stated otherwise, all topological notions refer to the group topology (except for boundedness which refers to the word metric  $d$ ).

We fix some compactly generated, locally compact Hausdorff group  $G$  and a word metric  $d$  on  $G$ . Notice that a subset of  $G$  is bounded with respect to  $d$  if and only if it is relatively compact (see Lemma A.1). Suppose that  $d'$  is another word metric on  $G$  or (more general) a metric which is quasi-isometrically equivalent to  $d$ . Then, by Theorem 3.3,  $t_{(G,d)}$  and  $t_{(G,d')}$  are bi-Lipschitz-equivalent. Hence linear equivalence and all notions which only depend on the topological or uniform structure of  $\mathcal{U}/\sim$  (like closure or Cauchy completeness for instance), do not depend on the generating set. In particular, we obtain the following statement.

**Lemma 5.3.** *If a compactly generated, locally compact Hausdorff group  $G$  is equipped with a word metric  $d$  then the spaces  $\mathcal{L}G$  and  $\mathcal{P}G$  do not depend on the choice of the word metric (or of the generating set).*

A group element  $g$  is called *compact*, if  $\langle g \rangle$  is relatively compact, and *non-compact* otherwise. Thus  $g$  is non-compact if and only if  $\langle g \rangle \in \mathcal{C}G$ . Notice that, if  $G$  is finitely generated, a group element  $g$  is non-compact if and only if  $g$  is non-torsion. Furthermore, by Weil's lemma (see [HR79, Theorem 9.1])  $g$  is non-compact, if and only if  $\langle g \rangle$  is the image of a monomorphism  $\mathbb{Z} \rightarrow G$  which is a topological isomorphism onto  $\langle g \rangle$  (a topological isomorphism is a group isomorphism which is also a homeomorphism). Hence, Weil's lemma implies the following.

**Lemma 5.4.** *If  $g \in G$  is non-compact then  $d(1, g^n) \rightarrow \infty$  for  $n \rightarrow \infty$ .*

*Remark.* Let  $g$  and  $h$  be non-compact. We have seen that  $s(\langle g \rangle^+, \langle h \rangle^+) \leq 1$  and  $s(\langle g \rangle^+, \langle g^n \rangle^+) = 0$  for all  $n \in \mathbb{N}$ . Now it is nearby to ask, what can be said about  $s(\langle g \rangle^+, \langle g^{-1} \rangle^+)$ . Often  $s(\langle g \rangle^+, \langle g^{-1} \rangle^+) = 1$ , but in [KLS12] Krön, Lehnert and Stein give an example of a finitely generated group constructed by iterated HNN-extensions with a non-torsion element  $g$  such that  $s(\langle g \rangle^+, \langle g^{-1} \rangle^+) \leq \frac{12}{17}$ . They also show that in general this value cannot be arbitrarily close to zero. Indeed,

414  $s(\langle g \rangle^+, \langle g^{-1} \rangle^+)$  is always greater or equal  $\frac{1}{2}$ . The infimum of these values (for all  
 415 groups) is unknown. In [KLS12] there is also an example of a finitely generated  
 416 group with non-torsion elements  $g, h$  for which  $\langle g \rangle^+ \sim \langle h \rangle^+$  but  $\langle g^{-1} \rangle^+ \not\sim \langle h^{-1} \rangle^+$ .

417 The following lemma yields a useful alternative to compute  $s^+(\langle g \rangle^+, \langle h \rangle^+)$  and  
 418  $s^+(\langle g \rangle, \langle h \rangle)$ .

419 **Lemma 5.5.** *Let  $g$  and  $h$  be non-compact group elements. Then*

$$s^+(\langle g \rangle^+, \langle h \rangle^+) = \limsup_{n \rightarrow \infty} \inf \left\{ \frac{d(h^n, g^m)}{d(1, g^m)} : m \in \mathbb{N}_0 \right\}$$

420 and

$$s^+(\langle g \rangle, \langle h \rangle) = \limsup_{|n| \rightarrow \infty} \inf \left\{ \frac{d(h^n, g^m)}{d(1, g^m)} : m \in \mathbb{Z} \right\}.$$

421 *Proof.* We only prove the first claim, since the proof of the second is analogous.  
 422 Suppose that  $\alpha > s^+(\langle g \rangle^+, \langle h \rangle^+)$ . Hence  $\langle h \rangle^+ \subseteq \alpha \langle h \rangle^+ + a$  for some  $a \geq 0$ . Thus,  
 423 for each  $n \in \mathbb{N}_0$ , there is an integer  $k = k(n) \geq 0$ , such that  $d(h^n, g^k) \leq \alpha d(1, g^k) + a$ .  
 424 Then

$$\inf \left\{ \frac{d(h^n, g^m)}{d(1, g^m)} : m \in \mathbb{N}_0 \right\} \leq \frac{d(h^n, g^k)}{d(1, g^k)} \leq \alpha + \frac{a}{d(1, g^k)}.$$

425 Using the triangle inequality we get

$$(1 - \alpha)d(1, g^k) - a \leq d(1, h^n) \leq (1 + \alpha)d(1, g^k) + a.$$

426 If  $n \rightarrow \infty$  then  $d(1, h^n) \rightarrow \infty$  by Lemma 5.4 and therefore  $d(1, g^k) \rightarrow \infty$ . This  
 427 implies

$$\limsup_{n \rightarrow \infty} \inf \left\{ \frac{d(h^n, g^m)}{d(1, g^m)} : m \in \mathbb{N}_0 \right\} \leq \limsup_{n \rightarrow \infty} \alpha + \frac{a}{d(1, g^k)} = \alpha.$$

428 In order to prove the reversed inequality assume that

$$\alpha > \limsup_{n \rightarrow \infty} \inf \left\{ \frac{d(h^n, g^m)}{d(1, g^m)} : m \in \mathbb{N}_0 \right\}.$$

429 Then there is an integer  $N \geq 0$ , such that

$$\inf \left\{ \frac{d(h^n, g^m)}{d(1, g^m)} : m \in \mathbb{N}_0 \right\} \leq \alpha$$

430 for all  $n \geq N$ . Let  $\varepsilon > 0$ . Then, for each  $n \geq N$ , we can find an integer  $k = k(n) \geq 0$ ,  
 431 such that

$$\frac{d(h^n, g^k)}{d(1, g^k)} \leq \alpha + \varepsilon.$$

432 Set  $a = \max\{d(1, h^n) : 0 \leq n < N\}$ . Then we obtain  $\langle h \rangle^+ \subseteq (\alpha + \varepsilon)\langle g \rangle^+ + a$ .  $\square$

433 Using Corollary A.6 and its notation, we obtain the following:

434 **Lemma 5.6.** *Let  $G$  be a compactly generated, locally compact group. The following*  
 435 *statements are true up to bi-Lipschitz-equivalence of the metric  $t$ :*

- 436 • *Suppose that  $N$  is a compact group and  $H$  is a topological (Hausdorff) group.*  
 437 *If  $\{1\} \longrightarrow N \longrightarrow H \xrightarrow{\pi} G \longrightarrow \{1\}$  is a topological short exact sequence,*  
 438 *such that  $\pi: H \rightarrow G$  is also open, then  $H$  and  $G$  have the same linear and*  
 439 *projective boundaries, respectively.*

- If  $H$  is a closed subgroup of  $G$  and  $(H \setminus G, d_{H \setminus G})$  is bounded then

$$\mathcal{L}H \subseteq \mathcal{L}G \quad \text{and} \quad \mathcal{P}H \subseteq \mathcal{L}G.$$

If  $H$  is of finite index in  $G$  then equality holds.

*Proof.* In order to prove the first statement note that, by Corollary A.6 the homomorphism  $\pi: H \rightarrow G$  is a quasi-isometry. Assume that  $h \in H$  and  $\langle \pi(h) \rangle$  is bounded in  $G$  then  $\pi^{-1}(\langle \pi(h) \rangle)$  is bounded by Lemma A.3. Hence  $\langle h \rangle \subseteq \pi^{-1}(\langle \pi(h) \rangle)$  is bounded. Thus unbounded cyclic sub(semi)groups of  $H$  are mapped onto unbounded cyclic sub(semi)groups of  $G$ . This implies the first statement using Theorem 3.3.

Now suppose that  $H$  is a closed subgroup of  $G$  and  $H \setminus G$  is bounded. By Corollary A.6 the inclusion is a quasi-isometry. In order to emphasize the dependence on  $H$  and  $G$ , we use subscripts  $H$  and  $G$ . By Theorem 3.3 we have

$$\mathcal{L}H = \text{cl}_H(\mathcal{C}^+H/\sim_H) = \text{cl}_G(\mathcal{C}^+H/\sim_G) \subseteq \text{cl}_G(\mathcal{C}^+G/\sim_G) = \mathcal{L}G$$

and analogously for  $\mathcal{P}H \subseteq \mathcal{P}G$ . Assume that  $H$  has finite index in  $G$ . If  $g \in G$  then  $H \setminus H\langle g \rangle$  is finite. Thus there are  $k \in \mathbb{Z}$  and  $n > 0$ , such that  $Hg^{k+n} = Hg^k$ . This implies  $g^n \in H$ . Hence in this case Lemma 5.2 implies

$$\mathcal{C}^+H/\sim_H = \mathcal{C}^+G/\sim_G \quad \text{and} \quad \mathcal{C}H/\sim_H = \mathcal{C}G/\sim_G$$

which yields the assertion.  $\square$

In the setting of finitely generated groups the previous lemma implies that two weakly commensurable finitely generated groups  $G$  and  $H$  (i.e. there is a group  $Q$  and homomorphisms  $Q \rightarrow G$  and  $Q \rightarrow H$  both having finite kernels and images of finite index) have the same linear and projective boundaries. In the continuous setting the situation is more complicated: In general it is possible that

$$\mathcal{C}^+H/\sim_H \subsetneq \mathcal{C}^+G/\sim_G$$

(consider for instance  $\mathbb{Z}^2 \leq \mathbb{R}^2$ ). However, equality may hold after taking closures on both sides, i.e.  $\mathcal{L}H = \mathcal{L}G$ . The problem here is to find for each non-compact  $g \in G$  a sequence  $(h_n)_{n \geq 0}$  in  $H$ , such that  $t(\langle g \rangle^+, \langle h_n \rangle^+) \rightarrow 0$  for  $n \rightarrow \infty$ . Notice that there is always an unbounded subset  $R \subseteq H$  with  $\langle g \rangle^+ \sim R$ , if  $H \setminus G$  is bounded.

With this preparations we can settle the commutative case completely. Recall that, if  $G$  is a commutative, compactly generated, locally compact Hausdorff group, then by [HR79, Theorem 9.8] there are integers  $a, b \geq 0$  and a commutative, compact Hausdorff group  $C$ , such that  $G$  is topologically isomorphic to  $\mathbb{R}^a \times \mathbb{Z}^b \times C$ .

**Corollary 5.7.** *Assume that  $G$  is a commutative, compactly generated, locally compact Hausdorff group.*

- If  $G$  is topologically isomorphic to  $\mathbb{R}^a \times \mathbb{Z}^b \times C$  for some integers  $a, b \geq 0$  and some compact, commutative group  $C$  then  $\mathcal{L}G = \mathbb{S}^{a+b-1}$  and  $\mathcal{P}G = \mathbb{P}^{a+b-1}$ .
- If  $H$  is a closed subgroup, such that  $G/H$  is compact then  $\mathcal{L}H = \mathcal{L}G$  and  $\mathcal{P}H = \mathcal{P}G$ .

*Proof.* As  $\mathbb{R}^a \times \mathbb{Z}^b$  is a quotient of  $G$  with compact kernel, the linear and projective boundaries of  $G$  and  $\mathbb{R}^a \times \mathbb{Z}^b$  are the same, respectively. Since any word metric on  $\mathbb{R}^a \times \mathbb{Z}^b$  is quasi-isometrically equivalent to the  $\ell^2$ -metric on  $\mathbb{R}^a \times \mathbb{Z}^b$ , we may use the  $\ell^2$ -metric. It is then easy to see that  $\mathbb{R}^a \times \mathbb{Z}^b$  and  $\mathbb{R}^{a+b}$  have the same boundaries. Hence the assertion follows from Example 2.11.

Suppose that  $H$  is a closed subgroup, such that  $G/H$  is compact. As before, let  $G$  be topologically isomorphic to  $\mathbb{R}^a \times \mathbb{Z}^b \times C$ . It follows that  $H$  is topologically isomorphic to  $\mathbb{R}^{a-c} \times \mathbb{Z}^{b+c} \times D$  for some integer  $c$  and some commutative, compact Hausdorff group  $D$ . Thus the first assertion implies the second.  $\square$

In the setting of topological groups it is nearby to consider also unbounded one-parameter subgroups and unbounded one-parameter subsemigroups, as well. A *one-parameter subgroup* in  $G$  is the image of a continuous homomorphism  $\mathbb{R} \rightarrow G$  and a *one-parameter subsemigroup* is the image of a continuous semigroup homomorphism  $[0, \infty) \rightarrow G$ . Obviously, if  $\varphi: [0, \infty) \rightarrow G$  is a continuous semigroup homomorphism, then there is a canonical extension to a continuous group homomorphism  $\bar{\varphi}: \mathbb{R} \rightarrow G$  and  $\varphi$  has unbounded image, if and only if  $\bar{\varphi}$  has. Define  $\mathcal{C}_{\mathbb{R}}G$  and  $\mathcal{C}_{\mathbb{R}}^+G$  to be the family of unbounded one-parameter subgroups and unbounded one-parameter subsemigroups, respectively. Again, by Weil's lemma a continuous homomorphism  $\varphi: \mathbb{R} \rightarrow G$  has unbounded image if and only if  $\varphi$  is a topological isomorphism onto its image.

**Lemma 5.8.** *Suppose that  $\varphi: \mathbb{R} \rightarrow G$  is a continuous homomorphism with unbounded image. Then*

$$\varphi([0, \infty)) \sim \langle \varphi(t) \rangle^+ \quad \text{and} \quad \varphi(\mathbb{R}) \sim \langle \varphi(t) \rangle$$

for all  $t > 0$ . Hence

$$\mathcal{C}_{\mathbb{R}}G/\sim \subseteq \mathcal{C}G/\sim \quad \text{and} \quad \mathcal{C}_{\mathbb{R}}^+G/\sim \subseteq \mathcal{C}^+G/\sim.$$

**Proposition 5.9.** *Let  $G$  be a connected, nilpotent Lie group. Then*

$$\mathcal{C}_{\mathbb{R}}G/\sim = \mathcal{C}G/\sim \quad \text{and} \quad \mathcal{C}_{\mathbb{R}}^+G/\sim = \mathcal{C}^+G/\sim.$$

*Proof.* Let  $\mathfrak{g}$  be the Lie algebra of  $G$  and  $\exp: \mathfrak{g} \rightarrow G$  be the exponential map. Then  $\exp$  is surjective. Thus, if  $g$  is a non-compact group element, then there is an element  $x \in \mathfrak{g}$  with  $\exp(x) = g$ . Then  $\mathbb{R} \rightarrow G, t \mapsto \exp(tx)$  is a continuous homomorphism with unbounded image which proves the statement.  $\square$

## 6. BOUNDARIES OF NILPOTENT GROUPS

In the following we determine the linear and projective boundary of connected, nilpotent Lie groups and their discrete counterparts, finitely generated nilpotent groups. A commutative, connected Lie group  $G$  is isomorphic to  $\mathbb{R}^a \times (\mathbb{R}/\mathbb{Z})^b$  for some integers  $a$  and  $b$ . In analogy to the discrete case we call the integer  $a$  the *compact-free dimension* of  $G$ . For convenience we define  $\mathbb{S}^{-1}$  and  $\mathbb{P}^{-1}$  to be the empty set.

**Theorem 6.1.** *Let  $G$  be a nilpotent group which is either a connected Lie group or a finitely generated group. Suppose that  $G$  has descending central series*

$$G = G_1 \supseteq G_2 \supseteq \cdots \supseteq G_c \supsetneq G_{c+1} = \{1\},$$

where  $c \geq 1$  is the nilpotency class of  $G$ . Let  $\nu(i)$  denote the compact-free dimension or torsion-free rank of  $G_i/G_{i+1}$ , respectively. Then the linear boundary  $\mathcal{L}G$  is homeomorphic to the disjoint union of  $c$  spheres:

$$\mathcal{L}G = \mathbb{S}^{\nu(1)-1} \uplus \mathbb{S}^{\nu(2)-1} \uplus \cdots \uplus \mathbb{S}^{\nu(c)-1}.$$

513 *Analogously, the projective boundary  $\mathcal{P}G$  is homeomorphic to the disjoint union of*  
 514 *projective spaces:*

$$\mathcal{P}G = \mathbb{P}^{\nu(1)-1} \uplus \mathbb{P}^{\nu(2)-1} \uplus \dots \uplus \mathbb{P}^{\nu(c)-1}.$$

515 If two finitely generated, nilpotent groups  $G$  and  $H$  are weakly commensurable  
 516 then previous result yields a new proof of the fact that the multisets

$$\{\nu_1(G), \nu_2(G), \dots\} \quad \text{and} \quad \{\nu_1(H), \nu_2(H), \dots\}$$

517 of torsion-free ranks are equal, since the boundaries of  $G$  and  $H$  are bi-Lipschitz-  
 518 equivalent. Notice that there is no information on the ordering and it is unclear,  
 519 whether it is possible to deduce the ordering from the angle metrics of  $G$  and  $H$ ,  
 520 respectively. It is a corollary of Pansu's theorem (see [Pan89, Théorème 3]), that  
 521 even the tuples

$$(\nu_1(G), \nu_2(G), \dots) \quad \text{and} \quad (\nu_1(H), \nu_2(H), \dots)$$

522 are equal.

523 First we prove the theorem for connected Lie groups and then use the Mal'tsev  
 524 completion to deduce the statement for finitely generated groups. For both cases  
 525 we use the notation and results of Appendix B.

526 *Proof of Theorem 6.1 in the Lie case.* We prove that statement for  $\mathcal{L}G$ , as the other  
 527 case is completely analogous. Let  $G$  be a connected, nilpotent Lie group with word  
 528 metric  $d_G$ . Set  $t_G = t_{(G, d_G)}$  and write  $\sim_G$  to denote linear equivalence in  $(G, d_G)$ .  
 529 By Lemma B.1 and Lemma 5.6 we may assume that  $G$  is also simply connected.  
 530 Set  $t_a = t_{(\mathfrak{g}, d_a)}$  and write  $\sim_a$  to denote linear equivalence in  $(\mathfrak{g}, d_a)$ . By the Lemmas  
 531 B.7, B.8, B.11 the map

$$\varphi: \mathcal{C}^+(\mathfrak{g}, +)/\sim_a \rightarrow \mathcal{C}^+(G, \cdot)/\sim_G$$

532 which maps the equivalence class of  $\langle x \rangle^+ \in \mathcal{C}^+(\mathfrak{g}, +)$  to the equivalence class of  
 533  $\langle \exp(x) \rangle^+ \in \mathcal{C}^+(G, \cdot)$ , is well-defined and bi-Hölder continuous with respect to the  
 534 metrics  $t_a$  and  $t_G$ , respectively. Hence  $\varphi$  extends to a bi-Hölder continuous map from  
 535  $\mathcal{L}(\mathfrak{g}, d_a)$  to  $\mathcal{L}(G, d_G)$ . Then the assertion follows from the first part of Lemma B.8.  
 536 □

537 *Proof of Theorem 6.1 in the discrete case.* Let  $\Gamma$  be a finitely generated, nilpotent  
 538 group. We only show the assertion for  $\mathcal{L}\Gamma$  for the same reason as above. By  
 539 Lemma B.1 and Lemma 5.6 we may assume that  $\Gamma$  is also torsion-free. Then the  
 540 (real) Mal'tsev completion of  $\Gamma$  yields a connected, simply connected, nilpotent Lie  
 541 group  $G$ , such that  $\Gamma$  is a uniform subgroup of  $G$ , see [Mal51]. Using Lemma 5.6  
 542 it follows that  $\mathcal{L}\Gamma \subseteq \mathcal{L}G$ . Let  $d_G$  be a word metric on  $G$  and set  $t_G = t_{(G, d_G)}$ .  
 543 In order to prove equality, it is sufficient to construct for each  $g \in G$  a sequence  
 544  $h_1, h_2, \dots \in \Gamma$ , such that  $t_G(\langle g \rangle^+, \langle h_m \rangle^+) \rightarrow 0$  if  $m \rightarrow \infty$ . Suppose that  $g$  is an  
 545 element of  $G_n$ . Set  $\Lambda = \log(\Gamma)$  and  $x = \log(g)$ . Then  $\Lambda \cap \mathfrak{g}_k$  is a uniform subgroup  
 546 in  $(\mathfrak{g}_k, \cdot)$  for all  $k$ . Hence  $\pi_n(\Lambda \cap \mathfrak{g}_n)$  is a uniform subgroup of  $(V_n, +)$ , since  $\pi_n$   
 547 is a continuous epimorphism from  $(\mathfrak{g}_n, \cdot)$  to  $(V_n, +)$ . As  $V_n$  is isomorphic to  $\mathbb{R}^{\nu(n)}$ ,  
 548  $\pi_n(\Lambda \cap \mathfrak{g}_n)$  is isomorphic to  $\mathbb{Z}^{\nu(n)}$ . Thus there is a sequence  $y_1, y_2, \dots \in \Lambda \cap \mathfrak{g}_n$ , such  
 549 that  $t_a(\langle \pi(x) \rangle^+, \langle \pi(y_m) \rangle^+) \rightarrow 0$  if  $m \rightarrow \infty$ . Since  $\langle x \rangle^+ \sim_a \langle \pi(x) \rangle^+$  and  $\langle y_m \rangle^+ \sim_a$   
 550  $\langle \pi(y_m) \rangle^+$ , we infer that  $t_a(\langle x \rangle^+, \langle y_m \rangle^+) \rightarrow 0$  if  $m \rightarrow \infty$ . Set  $h_m = \exp(z_m) \in \Gamma$ .  
 551 Then  $t_G(\langle g \rangle^+, \langle h_m \rangle^+) \rightarrow 0$  for  $m \rightarrow \infty$  using Lemma B.11 as required. □



552 *Remark.* We have carried out an alternative proof for the discrete case which avoids  
 553 the use of Mal'tsev completion and tools from Lie theory and employs techniques  
 554 from combinatorial group theory—mainly commutator calculus and careful analysis  
 555 of word lengths'. This proof follows similar lines compared to the proof for the Lie  
 556 case given here.

## 557 7. BOUNDARIES OF VERTEX-TRANSITIVE GRAPHS WITH POLYNOMIAL GROWTH

558 Let  $G$  be a group and let  $S$  be a finite generating set of  $G$ . Then the Cayley graph  
 559  $X$  of  $G$  with respect to  $S$  is given by  $VX = G$  and  $EX = \{\{g, gs\} : g \in G, s \in S\}$ .  
 560 If we define a Cayley graph in this way, namely by right multiplication, then  $G$  acts  
 561 as a vertex-transitive group of automorphisms on  $X$  by left multiplication. Hence  
 562 Cayley graphs of finitely generated groups give rise to a particular class of locally  
 563 finite, vertex-transitive graphs.

564 Since we have defined our notion of boundary for metric spaces in general, it is  
 565 nearby to consider  $\mathcal{LG}$  and  $\mathcal{PG}$  not only for groups  $G$  (and thus for their Cayley  
 566 graphs), but also for vertex-transitive graphs in general. But as Example 2.13 shows,  
 567 even for simple structures, as Cayley graphs of  $\mathbb{Z}^d$ , for Cayley graphs of groups  $G$  the  
 568 space  $\mathcal{U}/\sim$  is much richer than  $\mathcal{LG}$  or  $\mathcal{PG}$ . Hence it seems hopeless to characterize  
 569 our boundaries for graphs without involving group actions. Therefore, we define—  
 570 roughly speaking—the projective (linear) boundary of a graph as the projective  
 571 (linear) boundary induced by the action of its automorphism group. Then, at least  
 572 for graphs with polynomial growth, it is possible to obtain results similar to the  
 573 above.

574 Furthermore, we emphasize that the concepts defined in the sequel are not re-  
 575 stricted to locally finite graphs. In addition the results up to Corollary 7.8 also  
 576 hold without the assumption of local finiteness. From Theorem 7.9 to the end of  
 577 this section we consider graphs with polynomial growth which of course implies that  
 578 they are locally finite. Hence, although the main results of this section only hold for  
 579 locally finite graphs, this assumption is never explicitly stated.

580 In the following we always endow a graph  $X$  with the graph metric  $d$ , i.e., for any  
 581 two vertices  $u, v \in VX$ , the distance  $d(u, v)$  is the infimum of all numbers  $k$  such  
 582 that there is a path of length  $k$  connecting  $u$  and  $v$ .

583 **Definition 7.1.** Let  $X = (VX, EX)$  be an infinite, connected graph and let  $\text{Aut } X$   
 584 be the automorphism group of  $X$ . For  $v \in VX$ , we write  $\text{Unb}_v X \subseteq \text{Aut } X$  to denote  
 585 the set of group elements  $g \in \text{Aut } X$  for which the set  $\langle g \rangle v = \{g^n v : n \in \mathbb{Z}\}$  is  
 586 unbounded.

587 **Lemma 7.2.** Let  $X$  be an infinite, connected graph,  $v \in VX$ , and let  $g \in \text{Unb}_v X$ .

- 588 • The set  $\text{Unb}_v X$  is symmetric and both,  $g^\infty v = \{g^n v : n \in \mathbb{N}_0\}$  and  $(g^{-1})^\infty v =$   
 589  $\{g^{-n} v : n \in \mathbb{N}_0\}$ , are unbounded.
- 590 • If  $n$  is a nonzero integer then  $g^n \in \text{Unb}_v X$ . Furthermore,  $\langle g \rangle v$ ,  $\langle g^n \rangle v$  are  
 591 linear equivalent and  $g^\infty v$ ,  $(g^n)^\infty$  are linear equivalent, too.

592 *Proof.* The first part is immediate. The second part can be proved in the same way  
 593 as Lemma 5.2.  $\square$

594 **Definition 7.3.** Let  $X$  be an infinite, connected graph and  $G \leq \text{Aut } X$ . For  $v \in VX$   
 595 we define

$$\mathcal{C}_{G,v} X = \{\langle g \rangle v : g \in \text{Unb}_v X \cap G\}, \quad \mathcal{C}_{G,v}^+ X = \{g^\infty v : g \in \text{Unb}_v X \cap G\}.$$

**Lemma 7.4.** *Let  $X$  be an infinite, connected graph. Then, for  $u, v \in VX$ , we have*

$$\text{Unb}_u X = \text{Unb}_v X.$$

*If  $G \leq \text{Aut } X$  then*

$$\mathcal{C}_{G,u}X/\sim = \mathcal{C}_{G,v}X/\sim \quad \text{and} \quad \mathcal{C}_{G,u}^+X/\sim = \mathcal{C}_{G,v}^+X/\sim$$

*up to isometric isomorphy.*

*Proof.* Assume that  $g \in \text{Unb}_u X$ . Then  $d(u, g^n u) \rightarrow \infty$  as  $n \rightarrow \infty$ . As  $X$  is connected,  $d(u, v) < \infty$ . The triangle inequality implies

$$d(u, g^n u) \leq d(u, v) + d(v, g^n v) + d(g^n v, g^n u) = 2d(u, v) + d(v, g^n v),$$

hence  $d(v, g^n v) \geq d(u, g^n u) - 2d(u, v) \rightarrow \infty$  as  $n \rightarrow \infty$ . Thus  $\text{Unb}_u X \subseteq \text{Unb}_v X$  and the reversed inclusion follows by means of symmetry. In order to prove the second part of our assertion, note that, for  $g \in \text{Unb}_u X \cap G = \text{Unb}_v X \cap G$ ,

$$\langle g \rangle u \subseteq 0\langle g \rangle v + d(u, v) \quad \text{and} \quad \langle g \rangle v \subseteq 0\langle g \rangle u + d(u, v)$$

which means that  $\langle g \rangle u$  and  $\langle g \rangle v$  are linear equivalent, thus implying  $\mathcal{C}_{G,u}X/\sim = \mathcal{C}_{G,v}X/\sim$  up to isometric isomorphy. An analogous reasoning yields  $\mathcal{C}_{G,u}^+X/\sim = \mathcal{C}_{G,v}^+X/\sim$ .  $\square$

In the light of Lemma 7.4 we may drop dependence on the vertex  $v$ . This motivates the following definition:

**Definition 7.5.** Let  $X$  be an infinite, connected graph and fix a reference vertex  $v$ . Then we define  $\text{Unb } X = \text{Unb}_v X$ . If  $G$  is a subgroup of  $\text{Aut } X$  then we set

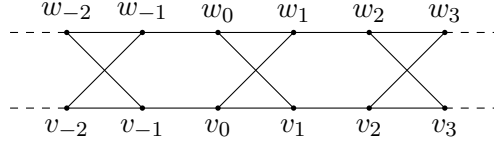
$$\mathcal{P}_G X = \text{cl}(\mathcal{C}_{G,v}X/\sim) \quad \text{and} \quad \mathcal{L}_G X = \text{cl}(\mathcal{C}_{G,v}^+X/\sim).$$

The spaces  $\mathcal{P}X = \mathcal{P}_{\text{Aut } X} X$  and  $\mathcal{L}X = \mathcal{L}_{\text{Aut } X} X$  are called *projective boundary* and *linear boundary* of  $X$ , respectively.

Let  $X$  be a graph and let  $\sigma$  be a partition of the vertex set  $VX$ . The quotient graph  $X_\sigma$  is defined as follows: the vertex set  $VX_\sigma$  is  $\sigma$ , and two vertices  $x, y \in VX_\sigma$  are adjacent, if there are adjacent vertices  $v, w \in X$  with  $v \in x$  and  $w \in y$ . Let  $G \leq \text{Aut } X$  be a group of automorphisms such that  $\sigma$  is  $G$ -invariant, i.e.  $g(b) \in \sigma$  for all  $b \in \sigma$  and all  $g \in G$ . Then  $G$  naturally induces a group action on  $X_\sigma$ . The subgroup of the automorphism group  $\text{Aut } X_\sigma$  corresponding to this action is denoted by  $G_\sigma$ . Also, there is a homomorphism  $\varphi: G \rightarrow G_\sigma$  such that the kernel of  $\varphi$  consists of all those  $g \in G$  with  $g(b) = b$  for all  $b \in \sigma$ . If  $G \leq \text{Aut } X$  acts vertex-transitively on  $X$  and  $\sigma$  is a  $G$ -invariant partition of  $VX$  then  $\sigma$  is called *imprimitivity system of  $G$  on  $X$* . The elements of an imprimitivity system are called *blocks*.

Let  $\sigma$  be an  $\text{Aut } X$ -invariant partition of  $VX$ . In order to avoid notational misunderstanding we write  $(\text{Aut } X)_\sigma$  to denote the subgroup of  $\text{Aut } X_\sigma$  corresponding to the natural action of  $\text{Aut } X$  on  $X_\sigma$ . Notice that  $(\text{Aut } X)_\sigma \subseteq \text{Aut } X_\sigma$ , but these two groups are not necessarily equal.

**Example 7.6.** Consider the graph  $X$  depicted in Figure 1. It consists of two disjoint infinite double-rays  $\{v_i : i \in \mathbb{Z}\}$  and  $\{w_i : i \in \mathbb{Z}\}$  and additional “crossed rungs”: For even  $i$ ,  $v_i$  is connected to  $w_{i+1}$  and for odd  $i$ ,  $v_i$  is connected to  $w_{i-1}$ . This graph is vertex-transitive, and the sets  $\{v_i, w_i\}$ ,  $i \in \mathbb{Z}$ , give rise to an imprimitivity system  $\sigma$  of  $\text{Aut } X$  on  $X$ . The quotient graph  $X_\sigma$  is an infinite double-ray  $\{x_i : i \in \mathbb{Z}\}$ , where the vertices  $x_i$  correspond to the sets  $\{v_i, w_i\}$  for  $i \in \mathbb{Z}$ . The mapping  $g_\sigma$


 FIGURE 1. An example graph  $X$  for  $(\text{Aut } X)_\sigma \subsetneq \text{Aut } X_\sigma$ .

633 which fixes  $x_0$  and maps  $x_i$  onto  $x_{-i}$  for  $i \in \mathbb{Z}$  is obviously an automorphism of  $X_\sigma$ .  
 634 But there exists no automorphism  $g \in \text{Aut } X$  with

$$g(\{v_i, w_i\}) = \{v_{-i}, w_{-i}\}$$

635 for  $i \in \mathbb{Z}$ . Hence, for this graph  $X$ ,  $(\text{Aut } X)_\sigma \subsetneq \text{Aut } X_\sigma$  holds.

636 Let  $X$  be an infinite, connected graph and  $H \leq G \leq \text{Aut } X$ . As the underlying  
 637 metric space  $(VX, d)$  is fixed, the inclusion  $H \leq G$  implies, that  $\mathcal{L}_H X$  and  $\mathcal{P}_H X$   
 638 are up to isometric isomorphism subspaces of  $\mathcal{L}_G X$  and  $\mathcal{P}_G X$ , respectively: Fix some  
 639 reference vertex  $v \in VX$  and notice that  $\mathcal{C}_{H,v}^+ \subseteq \mathcal{C}_{G,v}^+$ . Hence the map

$$\mathcal{C}_{H,v}^+ \rightarrow \mathcal{C}_{G,v}^+, \quad h^\infty v \mapsto h^\infty v$$

640 induces an isometric embedding  $\mathcal{C}_{H,v}^+ / \sim \rightarrow \mathcal{C}_{G,v}^+ / \sim$  which extends naturally to the  
 641 topological closures  $\mathcal{L}_H X$  and  $\mathcal{L}_G X$ . Similarly, there is an isometric embedding  
 642  $\mathcal{P}_H X \rightarrow \mathcal{P}_G X$ .

643 **Lemma 7.7.** *Let  $X$  be an infinite, connected graph.*

- 644 • *If  $H \leq G \leq \text{Aut } X$  and  $H$  has finite index in  $G$  then  $\mathcal{L}_H X$  and  $\mathcal{L}_G X$  ( $\mathcal{P}_H X$*   
 645 *and  $\mathcal{P}_G X$ ) are isometrically isomorphic.*
- 646 • *Let  $G \leq \text{Aut } X$  and let  $\sigma$  be a  $G$ -invariant partition of  $VX$  such that*

$$\sup\{d(x, y) : x, y \in b, b \in \sigma\} < \infty.$$

647 *Then  $\mathcal{L}_G X$  and  $\mathcal{L}_{G_\sigma} X_\sigma$  ( $\mathcal{P}_G X$  and  $\mathcal{P}_{G_\sigma} X_\sigma$ ) are bi-Lipschitz-equivalent.*

648 *Proof.* In order to prove the first statement, we may assume that  $H$  is a normal  
 649 subgroup of  $G$  with finite index, as the intersection of all conjugates of  $H$  forms  
 650 a normal subgroup with finite index. Let  $n$  be the finite index of  $H$  in  $G$ . Then,  
 651 for any  $g \in \text{Unb}_v X \cap G$ ,  $g^n \in \text{Unb}_v X \cap H$  and the unbounded subsets  $g^\infty v \in$   
 652  $\mathcal{C}_{G,v}^+$ ,  $(g^n)^\infty v \in \mathcal{C}_{H,v}^+$  are linearly equivalent. Therefore, the isometric embedding  
 653  $\mathcal{C}_{H,v}^+ / \sim \rightarrow \mathcal{C}_{G,v}^+ / \sim$  is an isometric isomorphism which extends naturally to  $\mathcal{L}_H X$  and  
 654  $\mathcal{L}_G X$ . An analogous reasoning yields the statement for  $\mathcal{P}_H X$  and  $\mathcal{P}_G X$ .

655 Let us now prove the second assertion. For  $x \in VX$  we write  $\bar{x}$  to denote the  
 656 element of  $\sigma = VX_\sigma$  containing  $x$ . Similarly, we write  $\bar{g} \in G_\sigma$  for the automorphism  
 657 of  $X_\sigma$  induced by the group element  $g \in G$ . Fix some reference vertex  $v$  and set

$$a = \sup\{d(x, y) : x, y \in b, b \in \sigma\} < \infty.$$

658 The map  $\pi: VX \rightarrow VX_\sigma$ ,  $x \mapsto \bar{x}$ , is a quasi-isometry, since

$$d_{X_\sigma}(\bar{x}, \bar{y}) \leq d_X(x, y) \leq (a + 1)d_{X_\sigma}(\bar{x}, \bar{y}) + a$$

659 for  $x, y \in VX$ . Furthermore,  $\pi$  induces a map from  $\mathcal{C}_{G,v}^+ X$  onto  $\mathcal{C}_{G_\sigma, \bar{v}}^+ X_\sigma$ : if  $g^\infty v =$   
 660  $\{v_0, v_1, \dots\} \in \mathcal{C}_{G,v}^+ X$  then  $\pi(g^\infty v) = \{\bar{v}_0, \bar{v}_1, \dots\} = \bar{g}^\infty \bar{v} \in \mathcal{C}_{G_\sigma, \bar{v}}^+ X_\sigma$ . Theorem 3.3  
 661 implies that

$$\mathcal{L}_G X = \text{cl}(\mathcal{C}_{G,v}^+ X / \sim) \quad \text{and} \quad \mathcal{L}_{G_\sigma} X_\sigma = \text{cl}(\mathcal{C}_{G_\sigma, \bar{v}}^+ X_\sigma / \sim)$$

are bi-Lipschitz-equivalent. Again the statement for  $\mathcal{P}_G X$  and  $\mathcal{P}_{G_\sigma} X_\sigma$  follows along the same lines.  $\square$

**Corollary 7.8.** *Let  $X$  be an infinite, connected graph.*

- *If  $G \leq \text{Aut } X$  acts vertex-transitively on  $X$  and  $\sigma$  is an imprimitivity system of  $G$  on  $X$  with finite blocks then  $\mathcal{L}_G X$  and  $\mathcal{L}_{G_\sigma} X_\sigma$  ( $\mathcal{P}_G X$  and  $\mathcal{P}_{G_\sigma} X_\sigma$ ) are bi-Lipschitz-equivalent.*
- *If  $G \leq \text{Aut } X$  acts freely and with finitely many orbits on  $VX$  then  $\mathcal{L}G$  and  $\mathcal{L}_G X$  ( $\mathcal{P}G$  and  $\mathcal{P}_G X$ ) are bi-Lipschitz-equivalent.*

*Proof.* The first statement is immediate:

$$\sup\{d(x, y) : x, y \in b, b \in \sigma\} < \infty$$

follows from the fact that  $G$  acts vertex-transitively on  $X$  and the blocks of  $\sigma$  are finite.

To prove the second statement we apply the ideas of the proof of the so-called Contraction Lemma (see [Bab77]): Since  $G$  acts freely and with finitely many orbits on  $X$ , there is a finite tree  $T$  in  $X$  which contains exactly one vertex of each orbit of  $G$  on  $X$ . Furthermore, the sets  $gVT$  for  $g \in G$  form a partition of  $VX$ . Set  $\sigma = \{gVT : g \in G\}$ . Then  $X_\sigma$  is isomorphic to a Cayley graph of  $G$ , and the groups  $G$  and  $G_\sigma$  are isomorphic, as  $VT$  contains exactly one vertex of each orbit. Hence  $\mathcal{L}G$  is (by definition) equal to  $\mathcal{L}_{G_\sigma} X_\sigma$  and the spaces  $\mathcal{L}_{G_\sigma} X_\sigma$ ,  $\mathcal{L}_G X$  are bi-Lipschitz-equivalent by the previous lemma.  $\square$

**Theorem 7.9.** *Let  $X$  be an infinite, connected, vertex-transitive graph with polynomial growth. Then there is a finitely generated, torsion-free, nilpotent group  $N$  which has the same growth rate as  $X$ , and  $\mathcal{L}N$  and  $\mathcal{P}N$  are bi-Lipschitz-equivalent to  $\mathcal{L}X$  and  $\mathcal{P}X$ , respectively.*

To prove this result about graphs with polynomial growth, the following two results of Trofimov [Tro84] are essential.

**Theorem 7.10** (Theorem 1 in [Tro84]). *Let  $X$  be an infinite, connected, vertex-transitive graph with polynomial growth. Then there exists an imprimitivity system  $\sigma$  of  $\text{Aut } X$  on  $VX$  with finite blocks such that  $\text{Aut } X_\sigma$  is a finitely generated virtually nilpotent group and the stabilizer in  $\text{Aut } X_\sigma$  of a vertex of  $X_\sigma$  is finite.*

**Theorem 7.11** (Theorem 2 in [Tro84]). *Let  $X$  be an infinite, connected graph with polynomial growth and let a group  $G \leq \text{Aut } X$  act vertex-transitively on  $VX$ . Then there exists an imprimitivity system  $\sigma$  of  $G$  on  $VX$  with finite blocks such that  $G_\sigma$  is a finitely generated virtually nilpotent group and the stabilizer in  $G_\sigma$  of a vertex of  $X_\sigma$  is finite.*

*Proof of Theorem 7.9.* Let  $G = \text{Aut } X$  and let  $\sigma$  and  $G_\sigma$  as in Theorem 7.11. Then  $G_\sigma$  contains a finitely generated, nilpotent, normal subgroup  $N$  of finite index. By [Sei91b, Corollary 2.7] we can furthermore assume that  $N$  is torsion-free. Since the finite index of  $N$  in  $G_\sigma$  implies that  $N$  acts with finitely many orbits on  $X$ , we can assume by [Sei91a, Theorem 2.3] that all  $n \in N$ ,  $n \neq 1$ , act with infinite orbits on  $X_\sigma$ .

Since the vertex stabilizers of  $\text{Aut } X_\sigma$  and  $G_\sigma$  are both finite (by Theorems 7.10 and 7.11), both groups have the same growth rate as the graph  $X_\sigma$  which is of course equal to the growth rate of  $X$ . Hence  $G_\sigma$  has finite index in  $\text{Aut } X_\sigma$ . As  $N$  has finite

index in  $G_\sigma$ , it has also finite index in  $\text{Aut } X_\sigma$ . Therefore Lemma 7.7 implies that the projective (linear) boundary induced by  $N$  on  $X_\sigma$  is bi-Lipschitz-equivalent to the projective (linear) boundary induced by  $\text{Aut } X_\sigma$  which we defined to be the projective (linear) boundary of  $X_\sigma$ .

Since  $X_\sigma$  is a quotient graph of  $X$  with respect to the finite blocks of  $\sigma$ , Corollary 7.8 implies that the projective (linear) boundaries of  $X$  and  $X_\sigma$  which are induced by  $\text{Aut } X$  and  $G_\sigma = (\text{Aut } X)_\sigma$ , respectively, are bi-Lipschitz-equivalent.

To conclude the proof we show that  $\mathcal{L}N$  and  $\mathcal{P}N$  are bi-Lipschitz-equivalent to  $\mathcal{L}_N X_\sigma$  and  $\mathcal{P}_N X_\sigma$ , respectively. As  $N$  is torsion-free and the stabilizer of a vertex is finite,  $N$  acts freely on  $X_\sigma$ . Since  $N$  also acts with finitely many orbits on  $X_\sigma$ , the claim follows directly from Corollary 7.8.  $\square$

As a consequence of Theorem 6.1 we obtain the following result.

**Corollary 7.12.** *Let  $X$  be an infinite, connected, vertex-transitive graph with polynomial growth and let  $N$  be a finitely generated, torsion-free, nilpotent group supplied by Theorem 7.9. Then the linear boundary  $\mathcal{L}X$  is homeomorphic to a disjoint union of spheres:*

$$\mathcal{L}X = \mathbb{S}^{\nu(1)-1} \uplus \mathbb{S}^{\nu(2)-1} \uplus \dots \uplus \mathbb{S}^{\nu(c)-1},$$

where  $c$  is the nilpotency class of  $N$  and  $\nu(i)$  is the torsion-free rank of the  $i$ -th quotient in the lower central series. Analogously, the projective boundary  $\mathcal{P}X$  is homeomorphic to a disjoint union of projective spaces:

$$\mathcal{P}X = \mathbb{P}^{\nu(1)-1} \uplus \mathbb{P}^{\nu(2)-1} \uplus \dots \uplus \mathbb{P}^{\nu(c)-1}.$$

Having these characterizations of the linear and projective boundaries of vertex-transitive graphs with polynomial growth, immediately the following question arises: When are the linear (projective) boundary of an infinite, connected, vertex-transitive graph  $X$  with polynomial growth and the linear (projective) boundary of its automorphism group  $\text{Aut } X$  bi-Lipschitz-equivalent? Using the concept of bounded automorphisms we are able to present a partial answer to this question.

An automorphism  $b \in \text{Aut } X$  is called *bounded* if there is an integer  $k$ , depending on  $b$ , such that  $d(x, b(x)) \leq k$  holds for all  $x \in VX$ . Of course the bounded automorphisms of  $X$  give rise to a normal subgroup  $B(X)$  of  $\text{Aut } X$ . As was shown in [GIS<sup>+</sup>89], the same holds for the bounded automorphisms of finite order of  $X$ . We denote the normal subgroup of  $\text{Aut } X$  generated by all bounded automorphisms of finite order by  $B_0(X)$ . As was also shown in [GIS<sup>+</sup>89],  $B_0(X)$  is locally finite, periodic and has finite orbits on  $X$ . Furthermore, in [Sei91b] the following result concerning  $B_0(X)$  was proved:

**Proposition 7.13** (Corollary 2.7 in [Sei91b]). *Let  $X$  be an infinite, connected graph with polynomial growth and let  $G \leq \text{Aut } X$  act vertex-transitively on  $X$ . Then the orbits of  $B_0(X) \cap G$  on  $X$  give rise to an imprimitivity system  $\sigma$  of  $G$  on  $VX$  such that  $G_\sigma$  satisfies the assertions of Theorem 7.11.*

Together with the following result of Sabidussi [Sab64], Proposition 7.13 now immediately implies a partial answer to the above formulated question. To formulate Sabidussi's result we need another definition.

If  $X$  is a graph and  $m$  is a cardinal then the graph  $mX$  is defined on the Cartesian product of  $VX$  by a set  $M$  of cardinality  $m$ , and

$$E(mX) = \left\{ \{(x, i), (y, j)\} : \{x, y\} \in EX, i, j \in M \right\}.$$

**Theorem 7.14** (Theorem 4 in [Sab64]). *Let  $X$  be a connected graph and let  $G \leq \text{Aut } X$  act vertex-transitively on  $X$ . Furthermore, let  $m$  denote the cardinality of the stabilizer in  $G$  of a vertex of  $X$ . Then  $mX$  is a Cayley graph of  $G$ .*

**Corollary 7.15.** *Let  $X$  be an infinite, connected, vertex-transitive graph with polynomial growth. Then  $\mathcal{L} \text{Aut } X$  and  $\mathcal{P} \text{Aut } X$  are bi-Lipschitz-equivalent to  $\mathcal{L}X$  and  $\mathcal{P}X$ , respectively, if  $B_0(X)$  is finite.*

*Proof.*  $B_0(X)$  is a normal subgroup of  $\text{Aut } X$ . If it is in addition finite then it follows from 7.13 and 7.10 that the stabilizer of a vertex of  $X$  in  $\text{Aut } X$  has some finite cardinality  $m$ . Then, by Theorem 7.14,  $mX$  is a Cayley graph of  $\text{Aut } X$  and arguments quite similar to those in the proof of Theorem 7.9 immediately complete the proof.  $\square$

In [Tro83] Trofimov defined a *lattice* as a connected locally finite graph  $X$ , such that for one of the groups  $G$ , acting vertex-transitively on  $X$ , there exists an imprimitivity system  $\sigma$  with finite blocks, such that  $G_\sigma$  is a finitely generated, commutative group. As was shown in [Tro83], in this case  $G \leq B(X)$  holds. Furthermore, it is obvious that lattices have polynomial growth with the same growth rate as  $G_\sigma$ . In addition lattices can be characterized as follows:

**Theorem 7.16** (Theorem 1 in [Tro83]). *Let  $X$  be a connected locally finite graph. Then  $X$  is a lattice if and only if a group  $G \leq B(X)$  acts vertex-transitively on  $X$ .*

This immediately leads to the following:

**Theorem 7.17.** *Let  $X$  be a connected locally finite graph of polynomial growth with growth rate  $r$  and let a group  $G \leq B(X)$  act vertex-transitively on  $X$ . Then*

$$\mathcal{L}X = \mathbb{S}^{r-1} \quad \text{and} \quad \mathcal{P}X = \mathbb{P}^{r-1}.$$

*Proof.* Applying Theorem 7.16 this result can be shown analogously to the proof of Theorem 7.9.  $\square$

Let  $X$  now be a Cayley graph of a group  $G$ . Then any group element  $g \in G$  gives rise to a bounded automorphism of  $X$  if and only if the conjugacy class of  $g$  in  $G$  is finite (see e.g. [GIS<sup>+</sup>89, page 335]). So the boundedness of an element  $g \in G$  is independent of whatever Cayley graph represents  $G$ .

A group  $G$  is called *FC-group* if for every  $g \in G$  the conjugacy class of  $g$  in  $G$  is finite. Hence for *FC*-groups  $G$  each  $g \in G$  acts as a bounded automorphism on any Cayley graph of  $G$ . Therefore Cayley graphs of finitely generated *FC*-groups are lattices and Theorem 7.17 immediately implies:

**Corollary 7.18.** *Let  $G$  be a finitely generated *FC*-group with polynomial growth of growth rate  $r$ . Then*

$$\mathcal{L}G = \mathbb{S}^{r-1} \quad \text{and} \quad \mathcal{P}G = \mathbb{P}^{r-1}.$$

781

## 8. ATTACHING THE BOUNDARY

Let  $\Xi$  be any subset of  $\mathcal{U}/\sim$ . In the following we describe a topology  $\tau$  on the disjoint union  $\bar{X}$  of  $X$  and  $\Xi$ , such that two requirements hold:

- The subspace topology of  $\tau$  on  $X$  is induced by the metric  $d$ .

784

785 • If  $x_1, x_2, \dots$  is a sequence in  $X$ , which eventually leaves any ball in  $X$ , and  
 786  $\xi$  is an equivalence class in  $\Xi$ , such that  $x_1, x_2, \dots \in R$  for some  $R \in \xi$  then  
 787  $x_1, x_2, \dots$  converges in  $\tau$  to  $\xi$ .

788 Due to the second requirement the subspace topology of  $\tau$  on  $\Xi$  is in general neither  
 789 induced by the metric  $t$  nor Hausdorff, see Lemma 8.2.

790 Fix some reference point  $o$  in  $X$  and let  $\xi \in \Xi$  be an equivalence class. If  $R \in \xi$   
 791 and  $\alpha > 0$  and  $r \geq 0$  then we set

$$N(R, \alpha, r) = \text{int}(\alpha R \setminus U(o, r)) \uplus \{\zeta \in \Xi : s^+(\xi, \zeta) < \alpha\}$$

792 where  $\text{int}(A)$  is the interior of the set  $A \subseteq X$ . Note that  $N(R, \alpha, p) \subseteq N(S, \beta, q)$  if  
 793  $R \subseteq S$ ,  $\alpha \leq \beta$ ,  $p \geq q$ . We define the topology  $\tau$  on  $\bar{X} = X \uplus \Xi$  by assigning to each  
 794  $x \in \bar{X}$  a family  $\mathcal{V}_x$  of sets which serves as an open neighborhood base for  $x$ :

- 795 • If  $x \in X$  then  $\mathcal{V}_x$  is the family of open balls centered at  $x$ .
- 796 • If  $\xi \in \Xi$  then  $\mathcal{V}_\xi$  is the family of sets  $N(R, \alpha, r)$  with  $R \in \xi$ ,  $\alpha > 0$ , and  
 797  $r \geq 0$ .

798 **Lemma 8.1.** *The families  $\mathcal{V}_x$ ,  $x \in \bar{X}$ , are open neighborhood bases of a topology  $\tau$*   
 799 *on  $\bar{X}$ . Its subspace topology on  $X$  is induced by the metric  $d$ ,  $X$  is dense and open*  
 800 *in  $\bar{X}$ , and the subspace topology on  $\Xi$  is  $T_0$ .*

801 *Proof.* By Theorem 4.5 in [Wil04] we have to check the following three conditions  
 802 for all  $x \in \bar{X}$ :

- 803 • If  $V \in \mathcal{V}_x$  then  $x \in V$ .
- 804 • If  $V_1, V_2 \in \mathcal{V}_x$  then  $V_3 \subseteq V_1 \cap V_2$  for some  $V_3 \in \mathcal{V}_x$ .
- 805 • If  $V \in \mathcal{V}_x$  and  $z \in V$  then  $W \subseteq V$  for some  $W \in \mathcal{V}_z$ .

806 The first condition is immediate for all  $x \in \bar{X}$  and the second and third condition  
 807 hold for all  $x \in X$ . Hence let  $\xi \in \Xi$ . In order to prove the second condition for  $\xi$   
 808 consider  $N(R, \alpha, p), N(S, \beta, q) \in \mathcal{V}_\xi$  with  $R, S \in \xi$ ,  $\alpha, \beta > 0$ , and  $p, q \geq 0$ . Choose  $\varepsilon$   
 809 in  $(0, \beta)$  and set

$$\gamma = \min\left\{\alpha, \frac{\beta - \varepsilon}{1 + \varepsilon}\right\}.$$

810 Since  $R, S \in \xi$ , it follows that  $s(R, S) = 0$  and by Lemma 2.5 there is a number  
 811  $r \geq \max\{p, q\}$  such that  $R \setminus U(o, r) \subseteq \varepsilon S$ . Using Lemma 2.2 this yields

$$\gamma R \subseteq \gamma(R \setminus U(o, r)) \cup \gamma U(o, r) \subseteq (\gamma + \varepsilon \gamma + \varepsilon) S \cup U(o, (1 + \gamma)r) \subseteq \beta S \cup U(o, (1 + \gamma)r)$$

812 by the choice of  $\gamma$ . Therefore

$$N(R, \gamma, (1 + \gamma)r) \subseteq N(R, \alpha, p) \cap N(S, \beta, q),$$

813 whence the second condition holds for  $\xi$ . The third condition holds for  $\xi$ , if  $z \in V \cap X$   
 814 or  $z = \xi$ . Hence consider  $V = N(R, \alpha, p)$  with  $R \in \xi$ ,  $\alpha > 0$ ,  $p \geq 0$ , and let  $\zeta \neq \xi$   
 815 be an element in  $V \cap \Xi$ . Choose an element  $S$  in  $\zeta$  and choose  $\beta$  in  $(s^+(R, S), \alpha)$ ,  
 816 which is possible, since  $s^+(R, S) = s^+(\xi, \zeta) < \alpha$ . There is a number  $r \geq p$ , such  
 817 that  $S \setminus U(o, r) \subseteq \beta R$ . Set  $\gamma = \frac{\alpha - \beta}{1 + \beta} > 0$ . Then

$$\gamma S \subseteq \gamma(S \setminus U(o, r)) \cup \gamma U(o, r) \subseteq (\gamma + \beta \gamma + \beta) R \cup U(o, (1 + \gamma)r) = \alpha R \cup U(o, (1 + \gamma)r)$$

818 by the choice of  $\beta$  and  $\gamma$ . Hence we obtain

$$N(S, \gamma, (1 + \gamma)r) \subseteq N(R, \alpha, p).$$

819 The last three assertions follow from the construction of  $\tau$ . □

820 *Remark.* Let  $X$  be an unbounded, locally compact, metric space. Then  $(\bar{X}, \tau)$  is  
 821 compact if the equivalence class of the unbounded set  $X$  is an element of  $\Xi$ . If,  
 822 apart from the equivalence class of  $X$ ,  $\Xi$  contains further elements then  $(\bar{X}, \tau)$  is  
 823 not Hausdorff.

824 **Lemma 8.2.** *Let  $\Xi$  be any subset of  $\mathcal{U}/\sim$  and let  $(\bar{X}, \tau)$  be defined as above.*

825 • *The space  $(\bar{X}, \tau)$  is Hausdorff if and only if*

$$s^+(\xi, \zeta) = 0 \iff s^+(\zeta, \xi) = 0$$

826 *for all  $\xi, \zeta \in \Xi$ . In this case, the subspace topology of  $\tau$  on  $\Xi$  is induced by*  
 827 *the metric  $t$ .*

828 • *Suppose that  $\Xi = \text{cl}(\mathcal{E}/\sim)$  for some family  $\mathcal{E} \subseteq \mathcal{U}$ . If there exists a func-*  
 829 *tion  $f: [0, 1] \rightarrow [0, \infty)$ , such that  $f(0) = 0$ ,  $f$  is continuous at 0, and*  
 830  *$s^+(S, R) \leq f(s^+(R, S))$  for all  $R, S \in \mathcal{E}$  then  $(\bar{X}, \tau)$  is Hausdorff and the*  
 831 *subspace topology of  $\tau$  on  $\text{cl}(\mathcal{E}/\sim)$  is induced by the metric  $t$ .*

832 *Proof.* The first assertion is a direct consequence of the definition of the open neigh-  
 833 borhood bases  $\mathcal{V}_\xi$  for  $\xi \in \Xi$ . The second statement is a consequence of the first,  
 834 since the hypotheses imply that

$$s^+(\xi, \zeta) = 0 \iff s^+(\zeta, \xi) = 0$$

835 for all  $\xi, \zeta \in \text{cl}(\mathcal{E}/\sim)$ : If  $s^+(\xi, \zeta) = 0$  and  $\varepsilon > 0$  is given then there are  $\xi', \zeta' \in \mathcal{E}/\sim$ ,  
 836 such that  $s(\xi, \xi') \leq \varepsilon$  and  $s(\zeta, \zeta') \leq \varepsilon$ . Thus

$$\begin{aligned} s^+(\zeta, \xi) &\leq 2\varepsilon + \varepsilon^2 + s^+(\zeta', \xi')(1 + \varepsilon)^2 \\ &\leq 2\varepsilon + \varepsilon^2 + f(s^+(\xi', \zeta'))(1 + \varepsilon)^2 \\ &\leq 2\varepsilon + \varepsilon^2 + f(2\varepsilon + \varepsilon^2)(1 + \varepsilon)^2. \end{aligned}$$

837 This shows that  $s^+(\zeta, \xi) = 0$ . □

838 With these preparations we are able to provide a criterion which ensures that  
 839 the topology defined above on the disjoint union of a compactly generated, locally  
 840 compact Hausdorff group  $G$  and its linear boundary  $\mathcal{L}G$  (projective boundary  $\mathcal{P}G$ )  
 841 is Hausdorff and the subspace topology on  $\mathcal{L}G$  ( $\mathcal{P}G$ ) is induced by the angle metric  
 842  $t$ .

843 **Proposition 8.3.** *Let  $G$  be a compactly generated, locally compact Hausdorff group.*  
 844 *Assume that there exists a constant  $C \geq 1$ , such that for every group element  $g \in G$*   
 845 *with  $\langle g \rangle^+ \in \mathcal{C}^+G$  there is an element  $\tilde{g} \in G$  with the following two properties:*

846 •  *$\langle \tilde{g} \rangle^+ \sim \langle g \rangle^+$  and*

847 •  *$d(1, \tilde{g}^m) \leq Cd(1, \tilde{g}^n) + C$  for all  $m, n$  with  $0 \leq m \leq n$ .*

848 *Then the topology  $\tau$  on  $G \uplus \mathcal{L}G$  defined by Lemma 8.1 is Hausdorff and the subspace*  
 849 *topology of  $\tau$  on  $\mathcal{L}G$  is induced by the metric  $t$ . An analogous statement holds for*  
 850 *the projective boundary.*

851 *Proof.* We check that the function  $f: [0, 1] \rightarrow [0, \infty)$ ,  $x \mapsto 2(1 + 4C)x$  satisfies the  
 852 conditions of the second part of Lemma 8.2 which implies the statement.

853 Of course,  $f$  is continuous and  $f(0) = 0$ . Furthermore, if  $x \geq \frac{1}{2}$ , then  $f(x) \geq$   
 854  $1 + 4C \geq 1$ . Hence  $s^+(\langle h \rangle^+, \langle g \rangle^+) \leq f(s^+(\langle g \rangle^+, \langle h \rangle^+))$  is trivially true if  $\langle g \rangle^+, \langle h \rangle^+ \in$   
 855  $\mathcal{C}^+G$  and  $s^+(\langle g \rangle^+, \langle h \rangle^+) \geq \frac{1}{2}$ , since  $s^+(\langle h \rangle^+, \langle g \rangle^+) \leq 1$ . Hence we may assume  
 856 that  $s^+(\langle g \rangle^+, \langle h \rangle^+) < \frac{1}{2}$ . Additionally, after replacing  $g$  by  $\tilde{g}$  if necessary, we may  
 857 assume that  $d(1, g^m) \leq Cd(1, g^n) + C$  for all  $m, n$  with  $m \leq n$ . Choose a number  $\alpha$



which satisfies  $s^+(\langle g \rangle^+, \langle h \rangle^+) < \alpha < \frac{1}{2}$ . Then there is a constant  $a \geq 0$ , such that  $\langle h \rangle^+ \subseteq \alpha \langle g \rangle^+ + a$ . Hence, for each  $n \in \mathbb{N}_0$  there is an integer  $\nu(n) \geq 0$  such that

$$d(h^n, g^{\nu(n)}) \leq \alpha d(1, g^{\nu(n)}) + a.$$

Now we define the function  $\kappa: \mathbb{N}_0 \rightarrow \mathbb{N}_0$  by

$$\kappa(n) = \min\{m \in \mathbb{N}_0 : \nu(m) \leq n \leq \nu(m+1)\}.$$

We claim that

$$d(g^n, h^{\kappa(n)}) \leq 2(1+4C)\alpha d(1, h^{\kappa(n)}) + 2Cd(1, h) + 2a + 8Ca + C$$

for all  $n \in \mathbb{N}_0$ . Once this claim is established then, by the second assertion of Lemma 8.2, the proof is finished. Let  $n \geq 0$  be an integer and set  $k = \kappa(n)$ . Since

$$d(g^n, h^k) \leq d(g^n, g^{\nu(k)}) + d(g^{\nu(k)}, h^k),$$

we need to find upper bounds for  $d(g^n, g^{\nu(k)})$  and  $d(g^{\nu(k)}, h^k)$ . See Figure 2 for a

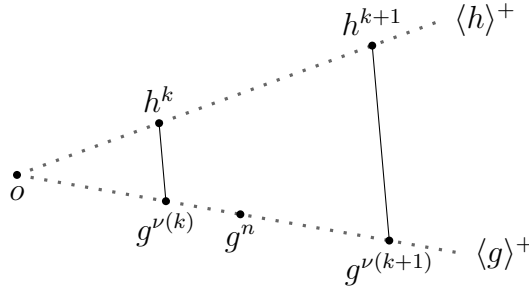


FIGURE 2. An outline of the situation of the proof

sketch of the constellation. Then

$$d(1, g^{\nu(k)}) \leq d(1, h^k) + d(h^k, g^{\nu(k)}) \leq d(1, h^k) + \alpha d(1, g^{\nu(k)}) + a$$

yields

$$d(1, g^{\nu(k)}) \leq \frac{1}{1-\alpha}(d(1, h^k) + a) \leq 2d(1, h^k) + 2a$$

using the bound  $\alpha \leq \frac{1}{2}$ . Thus

$$d(g^{\nu(k)}, h^k) \leq \alpha d(1, g^{\nu(k)}) + a \leq 2\alpha d(1, h^k) + 2a.$$

We obtain

$$\begin{aligned} d(g^{\nu(k)}, g^{\nu(k+1)}) &\leq d(g^{\nu(k)}, h^k) + d(h^k, h^{k+1}) + d(h^{k+1}, g^{\nu(k+1)}) \\ &\leq \alpha d(1, g^{\nu(k)}) + a + d(1, h) + \alpha d(1, g^{\nu(k+1)}) + a. \end{aligned}$$

Then  $d(1, g^{\nu(k+1)}) \leq d(1, g^{\nu(k)}) + d(g^{\nu(k)}, g^{\nu(k+1)})$  implies

$$d(g^{\nu(k)}, g^{\nu(k+1)}) \leq \alpha d(g^{\nu(k)}, g^{\nu(k+1)}) + 2\alpha d(1, g^{\nu(k)}) + d(1, h) + 2a$$

and by rearranging the last inequality we get

$$\begin{aligned} d(g^{\nu(k)}, g^{\nu(k+1)}) &\leq \frac{1}{1-\alpha}(2\alpha d(1, g^{\nu(k)}) + d(1, h) + 2a) \\ &\leq 4\alpha d(1, g^{\nu(k)}) + 2d(1, h) + 4a \\ &\leq 8\alpha d(1, h^k) + 2d(1, h) + 8a \end{aligned}$$

871 using the bound  $\alpha \leq \frac{1}{2}$  twice. The assumption on  $g$  implies

$$d(g^n, g^{\nu(k)}) \leq Cd(g^{\nu(k)}, g^{\nu(k+1)}) + C \leq 8C\alpha d(1, h^k) + 2Cd(1, h) + 8Ca + C.$$

872 Collecting the pieces yields

$$d(g^n, h^k) \leq 2(1 + 4C)\alpha d(1, h^k) + 2Cd(1, h) + 2a + 8Ca + C. \quad \square$$

873 **Lemma 8.4.** *Let  $G$  be a connected, nilpotent Lie group or a finitely generated,*  
874 *nilpotent group. Then the assumption of the previous proposition on  $G$  holds.*

875 *Proof.* Without loss of generality we may assume that  $G$  is simply connected in  
876 the Lie case or torsion-free in the discrete case, see Lemma B.1 and Corollary A.6.  
877 Furthermore, it is sufficient to prove the statement in the Lie case, as the discrete  
878 case follows by embedding  $G$  in its real Mal'tsev completion.

879 Hence suppose that  $G$  is a connected, simply connected, nilpotent Lie group and  
880 let  $d_G$  be a word metric on  $G$ . We use the notation of Appendix B. By Lemma B.6  
881 there exists a constant  $q$ , such that

$$q^{-1}|x| \leq d_G(1, \exp(x)) \leq q|x| + q$$

882 for all  $x \in \mathfrak{g}$ . We claim that the assumption of the previous proposition holds for  
883  $C = q^2$ . Let  $g$  be a group element of  $G$ . Then  $g \in G_i$  but  $g \notin G_{i+1}$  for some  
884  $i \geq 1$ . Set  $y = \pi_i(\log(g)) \in V_i$  and  $h = \exp(y)$ . Then, for  $0 \leq m \leq n$ , we have  
885  $|y^m| = m^{1/i}|y| \leq n^{1/i}|y| = |y^n|$  and therefore

$$d_G(1, h^m) \leq q|y^m| + q \leq q|y^n| + q \leq q^2 d_G(1, h^n) + q. \quad \square$$

## 886 9. RANDOM WALKS ON NILPOTENT GROUPS

887 Many aspects of random walks on nilpotent groups were studied, see for instance  
888 [Ale02, Gui73, Gui80, Kai91]. In the sequel we give a simple corollary of some results  
889 of Kaimanovich in [Kai91]. Let  $G$  be a connected, simply connected, nilpotent Lie  
890 group with descending central series

$$G = G_1 \supseteq G_2 \supseteq \dots G_c \supsetneq G_{c+1} = \{1\}$$

891 and let  $d_G$  be a word metric on  $G$ . A random walk  $(S_k)_{k \geq 0}$  on  $G$  has *finite first*  
892 *moment*, whenever

$$E(d_G(1, S_1)) = \int_G d_G(1, g) d\mu(g) < \infty,$$

893 where  $\mu$  is the law of  $S_1$ . Note that this notion does not depend on the choice of  
894 the word metric. We say that  $(S_k)_{k \geq 0}$  has *drift* if  $(S_k)_{k \geq 0}$  has finite first moment  
895 and there is an integer  $n \geq 1$ , such that  $S_1 \in G_n$  almost surely and  $(S_k G_{n+1})_{k \geq 0}$  is  
896 a random walk in the commutative group  $G_n/G_{n+1}$  with drift.

897 **Theorem 9.1.** *Let  $(S_k)_{k \geq 0}$  be a random walk with drift on a connected, simply*  
898 *connected Lie group  $G$ . Then there is a group element  $g$ , such that  $\{S_k : k \geq 0\} \sim$*   
899  *$\langle g \rangle^+$  holds almost surely. In terms of the topology on  $G \ltimes \mathcal{L}G$ , of Lemma 8.1, this*  
900 *means that almost surely  $(S_k)_{k \geq 0}$  converges to the equivalence class of  $\langle g \rangle^+$  in  $\mathcal{L}G$ .*  
901 *On the other hand, every point in  $\mathcal{L}G$  is limit point of a random walk with drift (in*  
902 *the sense above).*

903 *Proof.* Let  $n \geq 1$  be the integer, such that  $S_1 \in G_n$  almost surely and  $(S_k G_{n+1})_{k \geq 0}$   
 904 is a random walk with drift. By a result of Kaimanovich (see Theorem 4.2 and the  
 905 following Remark in [Kai91]) there is a group element  $g \in G_n$  ( $g \notin G_{n+1}$  by the  
 906 assumptions), such that  $d_{G_n}(S_k, g^k) = o(k)$  almost surely. This implies  $d_G(S_k, g^k) =$   
 907  $o(k^{1/n})$ . Since  $d_G(1, g^k) \geq C(g)k^{1/n}$  for some constant  $C(g) > 0$ , we obtain  $\{S_k :$   
 908  $k \geq 0\} \sim \langle g \rangle^+$  almost surely. And the other statements follow.

909 On the other hand, every point in  $\mathcal{L}G$  is limit point of a corresponding determin-  
 910 istic random walk.  $\square$

911 *Remark.* A similar statement holds for finitely generated, torsion-free, nilpotent  
 912 groups. Suppose that  $G$  is such a group and consider a random walk  $(S_k)_{k \geq 0}$  on  
 913  $G$  with drift. Then  $(S_k)_{k \geq 0}$  converges to an element in  $\mathcal{L}G$  with respect to the  
 914 topological space  $(G \uplus \mathcal{L}G, \tau)$ , where  $\tau$  is the topology of Lemma 8.1. On the other  
 915 hand, every point in  $\mathcal{L}G$  is limit point of a random walk with drift.

## 916 APPENDIX A. COMPACTLY GENERATED GROUPS

917 We provide some results on word metrics of compactly generated, locally compact  
 918 groups and related issues which are completely analogous to the case of finitely  
 919 generated groups. The books of Hewitt and Ross [HR79], Stroppel [Str06], and  
 920 de la Harpe [dlH00] provide a good background on topological and finitely generated  
 921 groups. We recall some basics from [Gui80].

922 **Lemma A.1** (Proposition 1 in [Gui80]). *Let  $G$  be a compactly generated, locally*  
 923 *compact Hausdorff group.*

- 924 • *If  $S$  is a compact, symmetric, generating set then, for some  $n \geq 0$ , the set*  
 925  *$S^n$  contains a neighborhood of 1.*
- 926 • *A subset of  $G$  is compact, if and only if it is closed and bounded with respect*  
 927 *to some word metric. Consequently, a subset is bounded if and only if it is*  
 928 *relatively compact.*
- 929 • *If  $S$  and  $S'$  are two compact symmetric generating sets then the associated*  
 930 *word metrics  $d$  and  $d'$  are bi-Lipschitz-equivalent, i.e. there is a constant*  
 931  *$q > 0$ , such that*

$$q^{-1}d(x, y) \leq d'(x, y) \leq qd(x, y)$$

932 *for all  $x, y \in G$ .*

933 *Proof.* For sake of completeness we provide a short proof: Since  $G$  is Hausdorff,  
 934 the sets  $S^n$ ,  $n = 0, 1, \dots$ , are closed and their union is equal to  $G$ . Hence, by  
 935 a well-known result on locally compact Hausdorff spaces (see for instance [Hoc65,  
 936 Lemma I.2.4]), there is an integer  $n \geq 0$ , such that  $S^n$  contains a non-empty open  
 937 subset. Since  $S^n$  is symmetric,  $S^{2n}$  contains a neighborhood of 1.

938 Let  $S$  be any compact, symmetric, generating subset of  $G$  and  $d$  be the associated  
 939 word metric. Choose  $n \geq 0$  such that  $S^n$  contains some open neighborhood  $U$  of 1.  
 940 Suppose that  $A$  is a compact subset of  $G$ . Then there are finitely many elements  
 941  $a_1, \dots, a_r$  of  $A$  such that  $A \subseteq a_1 U \cup \dots \cup a_r U$ . Thus

$$d(1, a) \leq n \max\{d(1, a_1), \dots, d(1, a_r)\}$$

942 for all  $a \in A$ . Hence  $A$  is bounded with respect to  $d$  and, since  $G$  is assumed to be  
 943 Hausdorff, the set  $A$  is also closed. Now suppose that  $A$  is a closed subset of  $G$  and

944 bounded with respect to  $d$ . Then  $A \subseteq S^m$  for some  $m \geq 0$  which implies that  $A$  is  
 945 compact.

946 By the second statement, there is a constant  $q > 0$ , such that  $d'(1, x) \leq q$  for all  
 947  $x \in S$  and  $d(1, y) \leq q$  for all  $y \in S'$ . This implies the third assertion.  $\square$

948 A metric space  $(X, d)$  is called  $q$ -quasi-geodesic, if for all  $x, y \in X$  there is an  
 949 integer  $n \geq 0$  and points  $x = x_0, x_1, \dots, x_{n-1}, x_n = y$  in  $X$ , such that

$$n \leq qd(x, y) + q \quad \text{and} \quad d(x_{i-1}, x_i) \leq q$$

950 for all  $1 \leq i \leq n$ . We remark that similar notions are used in the literature (see for  
 951 instance [BH99, Definition 8.22] and [Gro93, Section 0.2.D]). Of course, any geodesic  
 952 metric space is 1-quasi-geodesic and any word metric on a compactly generated,  
 953 locally compact group is 1-quasi-geodesic.

954 In the following we give a straight forward generalization of the classical Milnor-  
 955 Švarc lemma (see for instance [BH99, Proposition 8.19] or [dlH00, Theorem IV.B.23])  
 956 to the continuous case. Before stating the lemma we give a precise description of the  
 957 setting: Let  $G$  be a locally compact group and  $X$  be a Hausdorff space. Furthermore,  
 958 let  $d_X$  be a quasi-geodesic metric on  $X$  (we do not assume that  $d_X$  induces the  
 959 topology on  $X$ ). If not stated otherwise, all topological notions concerning  $X$  refer  
 960 to the topology on  $X$  with the exception of boundedness, which refers to the metric  
 961  $d_X$ . An action  $G \times X \rightarrow X$ ,  $(g, x) \mapsto gx$  is called

- 962 • *continuous*, if it is a continuous mapping from  $G \times X$  to  $X$ ,
- 963 •  *$q$ -cobounded*, if for all  $x, y \in X$  there is a  $g \in G$  with  $d_X(gx, y) \leq q$ ,
- 964 • *proper*, if  $\{g \in G : d_X(gx, x) \leq r\}$  is compact for all  $x \in X$  and all  $r \geq 0$ .

965 We say that  $G$  *acts by isometries*, if  $x \mapsto gx$  is an isometry with respect to  $d_X$  for  
 966 all  $g \in G$ . Note that if the action is continuous and  $K \subseteq G$  is compact then, for  
 967 any  $x \in X$ , the set  $Kx = \{gx : g \in K\}$  is compact and hence bounded. With these  
 968 preparations we are ready to state the lemma:

969 **Lemma A.2.** *Let  $G$  be a locally compact Hausdorff group and  $X$  be a Hausdorff*  
 970 *space which is additionally endowed with a quasi-geodesic metric  $d_X$ , such that all*  
 971 *compact subsets are bounded. Suppose that there is a continuous, cobounded, proper*  
 972 *action of  $G$  by isometries on  $X$ . Then  $G$  is compactly generated and for any  $x \in X$*   
 973 *the map  $G \rightarrow X$ ,  $g \mapsto gx$  is a quasi-isometry from  $(G, d_G)$  to  $(X, d_X)$ , where  $d_G$  is*  
 974 *some word metric of  $G$ .*

975 *Proof.* Except for minor modifications the proof is the same as in [BH99, dlH00].

976 For simplicity we assume that the constant  $q$  involved in the quasi-geodesic metric  
 977 is the same as the constant  $q$  of the cobounded action. Fix  $x \in X$ . Since the action  
 978 is proper, the set  $\{g \in G : d(gx, x) \leq 3q\}$  is compact. Let  $S$  be the union of this  
 979 set and its inverse. Then  $S$  is compact and symmetric and  $1 \in S$ .

980 We show that  $S$  generates  $G$ . Let  $g \in G$ . Since  $(X, d_X)$  is  $q$ -quasi-geodesic,  
 981 there are  $x = x_0, x_1, \dots, x_n = gx$ , such that  $n \leq qd(x, gx) + q$  and  $d(x_{i-1}, x_i) \leq q$   
 982 for  $1 \leq i \leq n$ . Since the action is  $q$ -cobounded, there are group elements  $g_0 =$   
 983  $1, g_1, \dots, g_n = g$ , such that  $d_X(g_i x, x_i) \leq q$  for all  $0 \leq i \leq n$ . Then

$$d_X(g_{i-1}^{-1} g_i x, x) = d_X(g_i x, g_{i-1} x) \leq d_X(g_i x, x_i) + d_X(x_i, x_{i-1}) + d_X(x_{i-1}, g_{i-1} x) \leq 3q.$$

984 It follows that  $s_i = g_{i-1}^{-1} g_i \in S$  and thus  $g = g_n = s_1 \cdots s_n \in S^n$ . Hence  $S$  is  
 985 a generating set. Let  $d_G$  be the word metric on  $G$  with respect to  $S$ . Then the

estimate above for  $g \in G$  yields

$$d_G(1, g) \leq n \leq qd_X(x, gx) + q.$$

Now we prove that  $G \rightarrow X$ ,  $g \mapsto gx$  is a quasi-isometry from  $(G, d_G)$  to  $(X, d_X)$ .  
Let  $g, h \in G$ . Then we obtain

$$d_G(g, h) = d_G(1, g^{-1}h) \leq qd_X(x, g^{-1}hx) + q = qd_X(gx, hx) + q.$$

For the reversed bound, note that  $Sx$  is bounded, since  $S$  is compact. Hence

$$M = \sup\{d_X(x, y) : y \in Sx\}$$

is finite. Suppose that  $d_G(g, h) = n \geq 1$  and  $g^{-1}h = s_1 \cdots s_n$  for some  $s_1, \dots, s_n \in S$ .  
Then

$$\begin{aligned} d_X(gx, hx) &= d_X(x, g^{-1}hx) = d_X(x, s_1 \cdots s_n x) \\ &\leq d_X(x, s_1 x) + d_X(s_1 x, s_1 s_2 x) + \cdots + d_X(s_1 \cdots s_{n-1} x, s_1 \cdots s_n x) \\ &= d_X(x, s_1 x) + d_X(x, s_2 x) + \cdots + d_X(x, s_n x) \\ &\leq Mn = Md_G(g, h). \end{aligned}$$

□

In order to have a handy reference we formulate the following well-known results,  
see [HR79, Section 5] and [Bou66, Section I.10.2].

**Lemma A.3.** *Let  $G$  be a Hausdorff group.*

- *Suppose that  $H$  is a subgroup. We write  $H \backslash G$  to denote the set of right cosets  $Hg$ ,  $g \in G$ , and equip  $H \backslash G$  with the quotient topology. Then the projection  $\pi: G \rightarrow H \backslash G$  is open (i.e. images of open sets are open). If  $H$  is compact then  $\pi$  is also proper (i.e. preimages of compact sets are compact).*
- *Suppose that  $\pi: H \rightarrow G$  is a continuous and open homomorphism which is onto. If the kernel of  $\pi$  is compact then  $\pi$  is proper.*

**Example A.4.** Let  $G$  be a compactly generated, locally compact Hausdorff group with word metric  $d_G$ ,  $N$  a compact Hausdorff group, and  $H$  a Hausdorff group. Suppose that

$$\{1\} \longrightarrow N \longrightarrow H \xrightarrow{\pi} G \longrightarrow \{1\}$$

is a topological exact sequence (i.e. all involved homomorphisms are continuous).  
The action  $H \times G \rightarrow G$ ,  $(h, g) \mapsto \pi(h)g$  is continuous and it acts by isometries. As  $\pi$  is onto, this action is obviously cobounded. Furthermore, the action is proper, if and only if

$$\{h \in H : d_G(hg, g) \leq r\} = \pi^{-1}(gB(1, r)g^{-1})$$

is compact for all  $g \in G$  and all  $r \geq 0$ . Here  $B(1, r)$  is the closed ball in  $G$  with respect to  $d_G$ . If  $\pi$  is an open map, it follows that the action is proper (Lemma A.3) and  $H$  is locally compact, since this is an extension property.

**Example A.5.** Consider a compactly generated, locally compact Hausdorff group  $G$  with word metric  $d_G$  and let  $H$  be a subgroup of  $G$ . Then  $H \times G \rightarrow G$ ,  $(h, g) \mapsto hg$  is a continuous action which acts by isometries. The set  $H \backslash G$  inherits a metric  $d_{H \backslash G}$  from  $G$ :

$$d_{H \backslash G}(Hg_1, Hg_2) = \min\{d_G(h_1g_1, h_2g_2) : h_1, h_2 \in H\}$$

for  $g_1, g_2 \in G$ , which is well-defined, since  $d_G$  is discrete. By left-invariance the action is cobounded, if and only if  $(H \backslash G, d_{H \backslash G})$  is bounded. Notice that  $(H \backslash G, d_{H \backslash G})$  is bounded, if  $H \backslash G$  is compact with respect to the quotient topology of  $G$ . To see

1018 this, choose  $n \geq 1$ , such that  $S^n$  contains an open neighborhood  $U$  of 1. Since the  
 1019 projection  $\pi: G \rightarrow H \backslash G$  is open (Lemma A.3),  $\{\pi(gU) : g \in G\}$  is an open cover  
 1020 of  $H \backslash G$ . Hence there is a finite subcover  $\{\pi(g_1U), \dots, \pi(g_mU)\}$ . Thus any coset of  
 1021  $H \backslash G$  is of the form  $Hg_iu$  for some  $1 \leq i \leq m$  and some  $u \in U$ . This yields the  
 1022 bound

$$\begin{aligned} d_{H \backslash G}(H, Hg_iu) &\leq d_G(1, g_iu) \leq d_G(1, g_i) + d_G(1, u) \\ &\leq \max\{d_G(1, g_i) : 1 \leq i \leq m\} + n. \end{aligned}$$

1023 If  $H$  is a closed subgroup then  $H$  is locally compact and this action is proper. To  
 1024 see this let  $g \in G$  and  $r \geq 0$  be given. Then

$$\{h \in H : d_G(hg, g) \leq r\} = gB(1, r)g^{-1} \cap H$$

1025 is compact, since  $gB(1, r)g^{-1}$  is compact and  $H$  is closed.

1026 By an application of the generalized Milnor-Švarc lemma to the situations de-  
 1027 scribed in the two previous examples we obtain the following:

1028 **Corollary A.6.** *Consider a compactly generated, locally compact Hausdorff group*  
 1029  *$G$  with word metric  $d_G$ .*

1030 • *Suppose that  $N$  is a compact Hausdorff group and  $H$  is a Hausdorff group*  
 1031 *and that*

$$\{1\} \longrightarrow N \longrightarrow H \xrightarrow{\pi} G \longrightarrow \{1\}$$

1032 *is a topological exact sequence, such that  $\pi: H \rightarrow G$  is open. Then  $H$  is com-*  
 1033 *pactly generated and locally compact and  $\pi$  is a quasi-isometry from  $(H, d_H)$*   
 1034 *to  $(G, d_G)$  for any word metric  $d_H$  on  $H$ .*

1035 • *If  $H$  is a closed subgroup of  $G$  and  $(H \backslash G, d_{H \backslash G})$  is bounded then  $H$  is com-*  
 1036 *pactly generated and locally compact and the inclusion is a quasi-isometry*  
 1037 *from  $(H, d_H)$  to  $(G, d_G)$  for any word metric  $d_H$  on  $H$ . Furthermore, if  $H \backslash G$*   
 1038 *is compact, then  $(H \backslash G, d_{H \backslash G})$  is bounded.*

1039 Finally, we note the following consequence of the Milnor-Švarc lemma, which says,  
 1040 that any reasonable metric on a compactly generated, locally compact Hausdorff  
 1041 group is quasi-isometrically equivalent to any word metric on the group.

1042 **Corollary A.7.** *Let  $G$  be a locally compact Hausdorff group. Suppose that  $d_Q$  is a*  
 1043 *left-invariant,  $q$ -quasi-geodesic metric on  $G$  with the property, that compact subsets*  
 1044 *are bounded with respect to  $d_Q$  and closed balls with respect to  $d_Q$  are compact. Then*  
 1045  *$G$  is compactly generated and  $d_Q$  is quasi-isometrically equivalent to any word metric*  
 1046 *on  $G$ .*

1047 Note that it is not assumed that the metric  $d_Q$  induces the group topology. How-  
 1048 ever, the assumptions guarantee some compatibility between the metric  $d_Q$  and the  
 1049 group topology. For example, the assumptions on  $d_Q$  are satisfied, if  $d_Q$  is left-  
 1050 invariant, geodesic, proper and induces the group topology.

## 1051 APPENDIX B. NILPOTENT LIE GROUPS

1052 The purpose of the appendix is to provide some background on nilpotent Lie  
 1053 groups, see for instance [CG90, Goo76, Hoc65], and some auxiliary results for The-  
 1054 orem 6.1.

1055 Let  $G$  be a group. We denote by  $[g, h] = g^{-1}h^{-1}gh$  the commutator in  $G$  and define  
 1056 the  $k$ -fold commutator inductively by  $[g_1] = g_1$  and  $[g_1, \dots, g_k] = [g_1, [g_2, \dots, g_k]]$ .  
 1057 The *descending central series* of  $G$  is inductively defined by

$$\gamma_1(G) = G \quad \text{and} \quad \gamma_{n+1}(G) = \langle [G, \gamma_n(G)] \rangle$$

1058 for  $n \geq 1$ . A group  $G$  is called *nilpotent* if  $\gamma_{n+1}(G) = \{1\}$  for some integer  $n$  and  
 1059 the least integer  $n$  with this property is called *nilpotency class* of  $G$ . If  $A$  is a subset  
 1060 of  $G$  then the set

$$I(A) = \{g \in G : g^n \in A \text{ for some } n \in \mathbb{N}\}$$

1061 is called *isolator* of  $A$ .

1062 If  $G$  is commutative and finitely generated, we denote its torsion-free rank by  
 1063  $\text{rk}(G)$ . If  $G$  is a commutative, connected Lie group then  $G$  is isomorphic to  $\mathbb{R}^a \times$   
 1064  $(\mathbb{R}/\mathbb{Z})^b$  for some integers  $a, b$ . In analogy to the discrete case we call  $a$  the *compact-*  
 1065 *free dimension* of  $G$  and denote it by  $\dim(G)$ .

1066 **Lemma B.1.** *Let  $G$  be a nilpotent group and set  $G_n = \gamma_n(G)$  for  $n \in \mathbb{N}$ .*

1067 • *If  $G$  is additionally a connected Lie group then the set  $C$  of all compact ele-*  
 1068 *ments in  $G$  is a characteristic, connected, compact subgroup,  $G/C$  is simply*  
 1069 *connected and*

$$\dim(\gamma_n(G/C)/\gamma_{n+1}(G/C)) = \dim(G_n/G_{n+1})$$

1070 *for all  $n \in \mathbb{N}$ .*

1071 • *If  $G$  is finitely generated then the set  $T$  of torsion elements in  $G$  is a char-*  
 1072 *acteristic, finite subgroup,  $G/T$  is torsion-free and*

$$\text{rk}(\gamma_n(G/T)/\gamma_{n+1}(G/T)) = \text{rk}(G_n/G_{n+1})$$

1073 *for all  $n \in \mathbb{N}$ .*

1074 • *If  $G$  is finitely generated and torsion-free then  $G = I(G_1) \supseteq I(G_2) \supseteq \dots$  is*  
 1075 *a central series of  $G$  with torsion-free quotients,  $G_n$  has finite index in  $I(G_n)$*   
 1076 *and*

$$\text{rk}(I_n(G)/I_{n+1}(G)) = \text{rk}(G_n/G_{n+1})$$

1077 *for all  $n \in \mathbb{N}$*

1078 *Proof.* Let  $G$  be a connected, nilpotent Lie group. Theorem 5.1 in [Glu55] implies  
 1079 the statements concerning  $C$  and  $G/C$ . It remains to show the equality concerning  
 1080 dimensions. By induction we have  $\gamma_n(G/C) = G_n C/C$  and it is easy to check that

$$G_n/G_{n+1} \rightarrow (G_n C/C)/(G_{n+1} C/C), \quad gG_{n+1} \mapsto gC \cdot (G_{n+1} C/C)$$

1081 is a continuous epimorphism with compact kernel which implies the equality.

1082 Now let  $G$  be a finitely generated, nilpotent group. Corollary 1.10 in [Seg83] yields  
 1083 the first part and the assertion concerning ranks follows mutatis mutandis.

1084 Finally, assume that  $G$  is a finitely generated, torsion-free, nilpotent group. By  
 1085 Lemma 3.4 in [Seg83]  $I(G_1) \supseteq I(G_2) \supseteq \dots$  is a central series with torsion-free quo-  
 1086 tients. Furthermore, it is easy to see that  $I(G_n)/G_n = T(G/G_n)$ , where  $T(G/G_n)$   
 1087 is the characteristic, finite subgroup of all torsion elements in  $G/G_n$ . Consider the  
 1088 map

$$G_n/G_{n+1} \rightarrow I(G_n)/I(G_{n+1}), \quad gG_{n+1} \mapsto gI(G_{n+1}).$$

1089 This is a homomorphism which has finite kernel and an image of finite index. This  
 1090 yields the claim concerning ranks.  $\square$

1091 In the following we fix a connected, simply connected, nilpotent Lie group  $G$  with  
 1092 nilpotency class  $c$  and set  $G_n = \gamma_n(G)$  for  $n \in \mathbb{N}$ . We denote by  $\mathfrak{g}$  the associated  
 1093 Lie algebra and by  $(x, y)$  the Lie bracket of  $\mathfrak{g}$ . Furthermore, we define the  $k$ -fold  
 1094 Lie bracket inductively by  $(x_1) = x_1$  and  $(x_1, \dots, x_k) = (x_1, (x_2, \dots, x_k))$ . The  
 1095 *descending central series* of  $\mathfrak{g}$  is

$$\mathfrak{g}_1 = \mathfrak{g} \quad \text{and} \quad \mathfrak{g}_{n+1} = \text{span}_{\mathbb{R}}(\mathfrak{g}, \mathfrak{g}_n)$$

1096 for  $n \geq 1$ . The Lie algebra of  $G_n$  is  $\mathfrak{g}_n$ . Let  $\nu(n)$  be the compact-free dimension of  
 1097  $G_n/G_{n+1}$ . Then

$$G_n/G_{n+1} \simeq \mathfrak{g}_n/\mathfrak{g}_{n+1} \simeq \mathbb{R}^{\nu(n)}$$

1098 as commutative groups. The *exponential map*  $\exp: \mathfrak{g} \rightarrow G$  is a diffeomorphism from  
 1099  $\mathfrak{g}$  to  $G$  and its inverse is  $\log: G \rightarrow \mathfrak{g}$ . The Baker-Campbell-Hausdorff formula yields  
 1100 a multiplicative group structure on  $\mathfrak{g}$ :

$$xy = x + y + \frac{1}{2}(x, y) + \frac{1}{12}(x, x, y) - \frac{1}{12}(y, x, y) - \frac{1}{24}(y, x, x, y) \pm \dots$$

1101 for  $x, y \in \mathfrak{g}$ . Then the exponential map  $\exp$  is a group isomorphism from  $(\mathfrak{g}, \cdot)$  to  
 1102  $(G, \cdot)$  and it is common to identify the Lie group  $G$  with its Lie algebra  $\mathfrak{g}$ .

1103 A subgroup  $\Gamma$  is called *uniform* in  $G$ , if  $\Gamma$  is discrete and the quotient  $\Gamma \backslash G$  is  
 1104 compact. In the following lemma we study uniform subgroups. Its proof depends  
 1105 on well-known results on such subgroups which can be found in [CG90, Chapter 5].

1106 **Lemma B.2.** *Let  $\Gamma$  be a uniform subgroup in  $G$  and set  $\Gamma_n = \gamma_n(\Gamma)$  for  $n \in \mathbb{N}$ .  
 1107 Then  $\Gamma \cap G_n = I(\Gamma_n)$  and*

$$\text{rk}(\Gamma_n/\Gamma_{n+1}) = \dim(G_n/G_{n+1})$$

1108 for all  $n \in \mathbb{N}$

1109 *Proof.* First we show that  $\Gamma \cap \gamma_n(G) = I(\gamma_n(\Gamma))$  for all  $n \in \mathbb{N}$  by backward induction  
 1110 on  $n$ :

- 1111 • Suppose that  $n = c$ : Obviously,  $I(\Gamma_c) \subseteq \Gamma$  and  $I(\Gamma_c) \subseteq G_c$ , hence  $I(\Gamma_c) \subseteq$   
 1112  $G_c \cap \Gamma$ . To prove the reversed inclusion, note that  $\exp$  is a group homo-  
 1113 morphism from  $(\mathfrak{g}_c, +)$  to  $(G_c, \cdot)$ . Let  $X \subseteq \mathfrak{g}$  be a strong Mal'tsev ba-  
 1114 sis strongly based on  $\Gamma$  and set  $Z = \exp(X)$ . Then  $\Gamma_c = \langle [Z, \dots, Z] \rangle$   
 1115 (see [MKS04, Theorem 5.4]) and thus  $\log(\Gamma_c) = \text{span}_{\mathbb{Z}}(X, \dots, X)$ , since  
 1116  $\exp((x_1, \dots, x_c)) = [\exp(x_1), \dots, \exp(x_c)]$  for all  $x_1, \dots, x_c \in \mathfrak{g}$ . Further-  
 1117 more, we have  $\mathfrak{g}_c = \text{span}_{\mathbb{R}}(X, \dots, X)$ . This implies that  $\Gamma_c$  and  $G_c \cap \Gamma$  are uni-  
 1118 form subgroups in  $G_c$ . Therefore  $(G_c \cap \Gamma)/\Gamma_c$  is finite, whence  $G_c \cap \Gamma \subseteq I(\Gamma_c)$ .  
 1119 • Assume that the claim holds for  $n \geq 2$ : Consider the groups  $G/G_n$  and  
 1120  $\Gamma G_n/G_n$ . Then  $\Gamma G_n/G_n$  is (topologically) isomorphic to  $\Gamma/(\Gamma \cap G_n)$ . By  
 1121  $\varphi$  we denote the canonical isomorphism  $\Gamma G_n/G_n \rightarrow \Gamma/(\Gamma \cap G_n)$ . Since  
 1122  $\Gamma G_n/G_n$  is a uniform subgroup in  $G/G_n$  and  $G/G_n$  is nilpotent with nilpo-  
 1123 tency class  $n - 1$ , using the initial step for the nilpotent group  $G/G_n$  yields

$$(\Gamma \cap G_{n-1})G_n/G_n = \Gamma G_n/G_n \cap \gamma_{n-1}(G/G_n) = I(\gamma_{n-1}(\Gamma G_n/G_n)).$$

1124 Applying the isomorphism  $\varphi$  on both sides we obtain

$$\begin{aligned} (\Gamma \cap G_{n-1})/(\Gamma \cap G_n) &= I(\gamma_{n-1}(\Gamma/(\Gamma \cap G_n))) \\ &= I(\Gamma_{n-1}(\Gamma \cap G_n))/(\Gamma \cap G_n) \\ &= I(\Gamma_{n-1})/(\Gamma \cap G_n) \end{aligned}$$



1125 using the induction hypothesis  $\Gamma \cap G_n = I(\Gamma_n)$  once more. It follows that  
 1126  $\Gamma \cap G_{n-1} = I(\Gamma_{n-1})$ .

1127 Now we prove the assertion concerning ranks. Since  $\Gamma \cap G_k$  is uniform in  $G_k$  for all  
 1128  $k \geq 1$ , it follows that  $(\Gamma \cap G_n)G_{n+1}/G_{n+1}$  is uniform in  $G_n/G_{n+1}$ . This implies that

$$\text{rk}((\Gamma \cap G_n)/(\Gamma \cap G_{n+1})) = \text{rk}((\Gamma \cap G_n)G_{n+1}/G_{n+1}) = \dim(G_n/G_{n+1})$$

1129 which yields the statement using the last part of Lemma B.1.  $\square$

1130 Since  $\mathfrak{g}$  is a real vector space of finite dimension  $\nu(1) + \dots + \nu(c)$ , there are linear  
 1131 subspaces  $V_n \subseteq \mathfrak{g}$  of dimension  $\nu(n)$ , such that  $\mathfrak{g}_n = V_n \oplus \mathfrak{g}_{n+1}$ . Hence

$$\mathfrak{g}_n = V_n \oplus \dots \oplus V_c.$$

1132 Write  $\pi_n: \mathfrak{g} \rightarrow V_n$  to denote the canonical projection. Then  $\pi_n$  is a continuous  
 1133 epimorphism from  $(\mathfrak{g}_n, \cdot)$  to  $(V_n, +)$  with kernel  $\mathfrak{g}_{n+1}$ . Let  $\|\cdot\|_n$  be some  $\ell^2$ -norm on  
 1134  $V_n$ . Then

$$\|x\| = \max\{\|\pi_n(x)\|_n : 1 \leq n \leq c\}$$

1135 is a norm on  $\mathfrak{g}$ . Notice that  $\|\pi_n(x)\| = \|\pi_n(x)\|_n$ . Since the Lie bracket  $(\cdot, \cdot)$  is  
 1136 bilinear, we have the following simple statement.

1137 **Lemma B.3.** *There is a constant  $M \geq 1$ , such that  $\|(x, y)\| \leq M\|x\| \|y\|$  for all*  
 1138  *$x, y \in \mathfrak{g}$ . Consequently,*

$$\|(x_1, \dots, x_k)\| \leq M^{k-1} \|x_1\| \dots \|x_k\|$$

1139 for all  $x_1, \dots, x_k \in \mathfrak{g}$ .

1140 For  $x \in \mathfrak{g}$  set

$$|x| = \max\{\|\pi_n(x)\|^{1/n} : 1 \leq n \leq c\}.$$

1141 Then  $|\cdot|$  is called (homogeneous) *gauge* or *quasi-norm* (see for instance [Bre12,  
 1142 Goo76, Gui73]). Note that  $|\cdot|$  is homogeneous with respect to the dilation  $\delta_t(x) =$   
 1143  $t\pi_1(x) + \dots + t^c\pi_c(x)$ , i.e.  $|\delta_t(x)| = t|x|$ , and it satisfies a weak form of the triangle  
 1144 inequality with respect to the Lie group structure on  $\mathfrak{g}$  (see Lemma B.5).

1145 **Lemma B.4.** *For all  $x, y \in \mathfrak{g}$  the following holds:*

- 1146 •  $|-x| = |x|$ ,
- 1147 •  $|x + y| \leq |x| + |y|$ ,
- 1148 • if  $x \in \mathfrak{g}_n$  and  $\alpha \geq 1$  then  $|\alpha x| \leq \alpha^{1/n} |x|$ ,
- 1149 • if  $0 \leq \alpha \leq 1$  then  $|\alpha x| \leq \alpha^{1/c} |x|$ .

1150 In any case,  $|\alpha x| \leq \max\{1, \alpha\} |x|$  for all  $\alpha \geq 0$ .

1151 The following lemma is a crucial observation due to Guivarc'h [Gui73, Lemme II.1],  
 1152 see also [Bre12, Lemma 2.5].

1153 **Lemma B.5.** *Let  $\alpha > 0$ . Then, by appropriately rescaling the norms  $\|\cdot\|_n$ , we have*

$$|xy| \leq |x| + |y| + \alpha$$

1154 for all  $x, y \in \mathfrak{g}$ .

1155 In the sequel we assume that the norms  $\|\cdot\|_n$  are chosen appropriately, so that the  
 1156 previous lemma holds with  $\alpha = 1$ . As a simple consequence we obtain  $|(x, y)| \leq$   
 1157  $2|x| + 2|y| + 2$  and it follows by induction, that

$$(2) \quad |(x_1, \dots, x_k)| \leq 2^{k-1}(|x_1| + \dots + |x_k|) + 2^k$$

1158 for all  $x_1, \dots, x_k \in \mathfrak{g}$ .

Since  $(G, \cdot) \simeq (\mathfrak{g}, \cdot)$  is a connected, locally compact group, it is compactly generated. Let  $d_w$  be some word metric on the group  $(\mathfrak{g}, \cdot)$ . The following result shows a fundamental connection between the gauge  $|\cdot|$  and the word metric  $d_w$ .

**Lemma B.6** (Theorem 2.7 in [Bre12]). *There is a constant  $q \geq 1$ , such that*

$$q^{-1}|x| \leq d_w(0, x) \leq q|x| + q$$

for all  $x \in \mathfrak{g}$ .

After providing the basic setup and important tools from Lie theory, we now apply the notions of Section 2 to this setting. We write  $s_w^+$  instead of  $s_{(\mathfrak{g}, d_w)}^+$ . The quantity  $d_a$  defined by  $d_a(x, y) = |-x + y|$  yields by Lemma B.4 a metric on  $\mathfrak{g}$ , and as before we write  $s_a^+$  instead of  $s_{(\mathfrak{g}, d_a)}^+$ . Although  $(x, y) \mapsto |x^{-1}y|$  is not a metric, we define

$$s_m^+(\langle x \rangle^+, \langle y \rangle^+) = \limsup_{n \rightarrow \infty} \inf \left\{ \frac{|y^{-n}x^m|}{|x^m|} : m \in \mathbb{N}_0 \right\}$$

and

$$s_m^+(\langle x \rangle, \langle y \rangle) = \limsup_{|n| \rightarrow \infty} \inf \left\{ \frac{|y^{-n}x^m|}{|x^m|} : m \in \mathbb{N}_0 \right\}$$

for  $x, y \in \mathfrak{g} \setminus \{0\}$ . Using Lemma 5.5 and Lemma B.6 we get the following comparison of  $s_w^+$  and  $s_m^+$ .

**Lemma B.7.** *Let  $x, y \in \mathfrak{g}$  with  $x \neq 0$  and  $y \neq 0$ . Then*

$$q^{-2}s_m^+(\langle x \rangle^+, \langle y \rangle^+) \leq s_w^+(\langle x \rangle^+, \langle y \rangle^+) \leq q^2s_m^+(\langle x \rangle^+, \langle y \rangle^+)$$

and

$$q^{-2}s_m^+(\langle x \rangle, \langle y \rangle) \leq s_w^+(\langle x \rangle, \langle y \rangle) \leq q^2s_m^+(\langle x \rangle, \langle y \rangle),$$

where  $q$  is the constant of Lemma B.6.

Our goal is the comparison of  $s_a^+$  and  $s_m^+$ . We restrict this comparison to elements of  $\mathcal{C}^+\mathfrak{g}$  and  $\mathcal{C}\mathfrak{g}$ . Note that  $\mathcal{C}^+(\mathfrak{g}, \cdot) = \mathcal{C}^+(\mathfrak{g}, +)$  and  $\mathcal{C}(\mathfrak{g}, \cdot) = \mathcal{C}(\mathfrak{g}, +)$ , since  $x^n = nx$  for all  $x \in \mathfrak{g}$  and  $n \in \mathbb{Z}$ . Before we provide the necessary tools for this comparison, let us identify  $\mathcal{L}(\mathfrak{g}, d_a)$  and  $\mathcal{P}(\mathfrak{g}, d_a)$ .

**Lemma B.8.** *Up to homeomorphism we have*

$$\mathcal{L}(\mathfrak{g}, d_a) = \mathbb{S}^{\nu(1)-1} \uplus \dots \uplus \mathbb{S}^{\nu(c)-1}, \quad \mathcal{P}(\mathfrak{g}, d_a) = \mathbb{P}^{\nu(1)-1} \uplus \dots \uplus \mathbb{P}^{\nu(c)-1}.$$

Moreover, the following three statements yield a precise description of  $\mathcal{L}(\mathfrak{g}, d_a)$  and  $\mathcal{P}(\mathfrak{g}, d_a)$ .

(a) *If  $x, y \in \mathfrak{g}_i$  and  $x + \mathfrak{g}_{i+1} = y + \mathfrak{g}_{i+1} \neq \mathfrak{g}_{i+1}$  then*

$$s_a^+(\langle x \rangle^+, \langle y \rangle^+) = 0 \quad \text{and} \quad s_a^+(\langle x \rangle, \langle y \rangle) = 0.$$

(b) *If  $x \in \mathfrak{g}_i$ ,  $x \notin \mathfrak{g}_{i+1}$ , and  $y \in \mathfrak{g}_{i+1}$  then*

$$s_a^+(\langle x \rangle^+, \langle y \rangle^+) = 1 \quad \text{and} \quad s_a^+(\langle x \rangle, \langle y \rangle) = 1.$$

(c) *If  $x, y \in V_i$  and  $x, y \neq 0$  then, using the notation of Example 2.11,*

$$s_a^+(\langle x \rangle^+, \langle y \rangle^+) = (\sin(\min\{\frac{1}{2}\pi, \angle(H_x, H_y)\}))^{1/i}$$

and

$$s_a^+(\langle x \rangle, \langle y \rangle) = (\sin(\angle(L_x, L_y)))^{1/i}.$$

1185 *Proof.* Once we have proved (a), (b), (c) the statement of the lemma follows. We  
 1186 only prove these three statements for  $s_a^+(\langle x \rangle^+, \langle y \rangle^+)$  the other case being analogous.  
 1187 *Statement (a).* By assumption  $-y + x \in \mathfrak{g}_{i+1}$ , whence

$$|-ny + nx| = |n(-y + x)| \leq n^{1/(i+1)} |-y + x|.$$

1188 Since  $x \in \mathfrak{g}_i \setminus \mathfrak{g}_{i+1}$ , it follows that  $\pi_i(x) \neq 0$  and

$$|nx| \geq |\pi_i(nx)| = n^{1/i} |\pi_i(x)|.$$

1189 From this we infer that

$$s_a^+(\langle x \rangle^+, \langle y \rangle^+) \leq \limsup_{n \rightarrow \infty} \frac{|-ny + nx|}{|nx|} \leq \limsup_{n \rightarrow \infty} \frac{n^{1/(i+1)} |-y + x|}{n^{1/i} |\pi_i(x)|} = 0.$$

1190 *Statement (b).* Using (a), we may assume that  $x \in V_i$ . Then  $\pi_i(-ny + mx) = mx$   
 1191 and so

$$|-ny + mx| \geq |\pi_i(-ny + mx)| = |mx|.$$

1192 This implies

$$\inf \left\{ \frac{|-ny + mx|}{|mx|} : m \in \mathbb{N}_0 \right\} \geq 1$$

1193 and therefore  $s_a^+(\langle x \rangle^+, \langle y \rangle^+) \geq 1$ .

1194 *Statement (c).* Note that  $|v| = \|v\|^{1/i}$  for all  $v \in V_i$ . Since  $s_a^+(\langle x \rangle^+, \langle y \rangle^+) =$   
 1195  $s_a^+(H_x, H_y)$ , the statement follows from Example 2.11.  $\square$

1196 We now compare  $s_a^+$  and  $s_w^+$ . Let  $y, z$  be elements in  $\mathfrak{g}$  and consider the product  
 1197  $y^{-1}(y + z) = (-y)(y + z)$ . Then, using the Baker-Campbell-Hausdorff formula,

$$\begin{aligned} y^{-1}(y + z) &= (-y) + (y + z) + \frac{1}{2}(-y, y + z) + \frac{1}{12}(-y, -y, y + z) \\ &\quad - \frac{1}{12}(y + z, -y, y + z) \pm \cdots \\ (3) \quad &= z - \frac{1}{2}(y, z) + \frac{2}{12}(y, y, z) + \frac{1}{12}(z, y, z) \pm \cdots. \end{aligned}$$

1198 Of course in the last expression above at most  $c$ -fold Lie brackets occur and, for each  
 1199  $1 \leq k \leq c$ , there are finitely many  $k$ -fold Lie brackets, say  $v_{k,1}, \dots, v_{k,m(k)}$ , whose  
 1200 entries are either  $y$  or  $z$ , and each of which contains at least one  $y$  and at least one  
 1201  $z$ . If  $1 \leq k \leq c$  and  $1 \leq j \leq m(k)$  then write  $q_{k,j}$  for the rational coefficient in front  
 1202 of the  $k$ -fold Lie bracket  $v_{k,j}$ . Then

$$y^{-1}(y + z) = \sum_{1 \leq k \leq c} \sum_{1 \leq j \leq m(k)} q_{k,j} v_{k,j}.$$

1203 Note that the constants  $q_{k,j}$  depend on the Baker-Campbell-Hausdorff formula only.  
 1204 For convenience we set  $Q_{k,j} = \max\{1, q_{k,j}\}$  and

$$Q = \sum_{1 \leq k \leq c} \sum_{1 \leq j \leq m(k)} Q_{k,j}.$$

1205 **Lemma B.9.** *Suppose that  $x, y \in \mathfrak{g}_i$  and  $x\mathfrak{g}_{i+1} = y\mathfrak{g}_{i+1} \neq \mathfrak{g}_{i+1}$ . Then*

$$|y^{-n}x^n| \leq 2^{c-1}Q(c|x| + c|y| + 2)n^{(1-1/c)/i}$$

1206 *for all  $n \geq 0$ .*

1207 *Proof.* Set  $z = x - y$  and  $m = |x| + |y|$ . By assumption  $z \in \mathfrak{g}_{i+1}$  and obviously  
 1208  $|x|, |y|, |z| \leq m$ . Using the representation (3) of the product  $y^{-1}(y + z)$  we obtain

$$y^{-n}x^n = y^{-n}(y + z)^n = \sum_{1 \leq k \leq c} \sum_{1 \leq j \leq m(k)} q_{k,j} n^k v_{k,j}.$$

1209 Since each  $k$ -fold Lie bracket  $v_{k,j}$  contains at least one  $z$ , we get  $v_{k,j} \in \mathfrak{g}_{ki+1}$ . Using  
 1210 (2) yields  $|v_{k,j}| \leq 2^{k-1}km + 2^k = 2^{k-1}(km + 2)$  for all  $k, j$  and therefore

$$|q_{k,j} n^k v_{k,j}| \leq Q_{k,j} n^{k/(ki+1)} 2^{k-1}(km + 2).$$

1211 Collecting the pieces, we obtain

$$\begin{aligned} |y^{-n}x^n| &\leq \sum_{1 \leq k \leq c} \sum_{1 \leq j \leq m(k)} |q_{k,j} n^k v_{k,j}| \\ &\leq \sum_{1 \leq k \leq c} \sum_{1 \leq j \leq m(k)} Q_{k,j} n^{k/(ki+1)} 2^{k-1}(km + 2) \\ &\leq 2^{c-1} Q(cm + 2) n^{(1-1/c)/i} \quad \square \end{aligned}$$

1212 **Lemma B.10.** Suppose that  $x, y \in V_i$  and  $|x| \geq |y| = 1$  and  $|x - y| = \alpha|x|$  for  
 1213 some  $\alpha \in [0, 1]$ . Then

$$|y^{-n}x^n| \leq MQ\alpha^{i/c}|x^n|$$

1214 for all  $n \geq 0$ .

1215 *Proof.* Set  $z = x - y \in V_i$ . Of course  $\|x\| \geq \|y\| = 1$ , and  $\|z\| = \alpha^i\|x\|$ . Using the  
 1216 representation (3) we get as in the proof above

$$y^{-n}x^n = y^{-n}(y + z)^n = \sum_{1 \leq k \leq c} \sum_{1 \leq j \leq m(k)} q_{k,j} n^k v_{k,j}.$$

1217 Each  $k$ -fold Lie bracket  $v_{k,j}$  contains at least on  $z$ , but this time  $v_{k,j} \in \mathfrak{g}_{ki}$ . An  
 1218 application of Lemma B.3 implies

$$\begin{aligned} |v_{k,j}| &= \max\{\|\pi_j(v_{k,j})\|^{1/l} : ik \leq l \leq c\} \\ &\leq \max\{\|v_{k,j}\|^{1/l} : ik \leq l \leq c\} \\ &\leq \max\{(M^{k-1}\alpha^i\|x\|^k)^{1/l} : ik \leq l \leq c\} \\ &\leq M\alpha^{i/c}\|x\|^{1/i} = M\alpha^{i/c}|x|. \end{aligned}$$

1219 Hence we obtain

$$\begin{aligned} |y^{-n}x^n| &\leq \sum_{1 \leq k \leq c} \sum_{1 \leq j \leq m(k)} |q_{k,j} n^k v_{k,j}| \\ &\leq \sum_{1 \leq k \leq c} \sum_{1 \leq j \leq m(k)} Q_{k,j} n^{1/i} M\alpha^{i/c}|x| \\ &= MQ\alpha^{i/c}|x^n| \quad \square \end{aligned}$$

1220 **Lemma B.11.** The following three statements hold.

1221 (a) If  $x, y \in \mathfrak{g}_i$  and  $x\mathfrak{g}_{i+1} = y\mathfrak{g}_{i+1} \neq \mathfrak{g}_{i+1}$  then

$$s_m^+(\langle x \rangle^+, \langle y \rangle^+) = 0 \quad \text{and} \quad s_m^+(\langle x \rangle, \langle y \rangle) = 0.$$

1222 (b) If  $x \in \mathfrak{g}_i$ ,  $x \notin \mathfrak{g}_{i+1}$ , and  $y \in \mathfrak{g}_{i+1}$  then

$$s_m^+(\langle x \rangle^+, \langle y \rangle^+) = 1 \quad \text{and} \quad s_m^+(\langle x \rangle, \langle y \rangle) = 1.$$

1223 (c) If  $x, y \in V_i$  and  $x, y \neq 0$  then

$$s_a^+(\langle x \rangle^+, \langle y \rangle^+) \leq s_m^+(\langle x \rangle^+, \langle y \rangle^+) \leq MQ(s_a^+(\langle x \rangle^+, \langle y \rangle^+))^{i/c}$$

1224 and

$$s_a^+(\langle x \rangle, \langle y \rangle) \leq s_m^+(\langle x \rangle, \langle y \rangle) \leq MQ(s_a^+(\langle x \rangle, \langle y \rangle))^{i/c}.$$

1225 *Proof.* *Statement* (a). By assumption  $\pi_i(x) \neq 0$  and we get

$$|x^n| \geq |\pi_i(x^n)| = n^{1/i} |\pi_i(x)|.$$

1226 On the other hand Lemma B.9 implies

$$|y^{-n}x^n| \leq 2^{c-1}Q(c|x| + c|y| + 2)n^{(1-1/c)/i}$$

1227 for all  $n \geq 0$ . Hence

$$\begin{aligned} s_m^+(\langle x \rangle^+, \langle y \rangle^+) &\leq \limsup_{n \rightarrow \infty} \frac{|y^{-n}x^n|}{|x^n|} \\ &\leq \limsup_{n \rightarrow \infty} \frac{2^{c-1}Q(c|x| + c|y| + 2)n^{(1-1/c)/i}}{n^{1/i}|\pi_i(x)|} = 0. \end{aligned}$$

1228 *Statement* (b): By the first claim we may assume that  $x \in V_i$ . Using the Baker-  
1229 Campbell-Hausdorff formula we obtain  $\pi_i(y^{-n}x^m) = x^m$  and thus

$$|y^{-n}x^m| \geq |\pi_i(y^{-n}x^m)| = |x^m|.$$

1230 This implies

$$\inf \left\{ \frac{|y^{-n}x^m|}{x^m} : m \in \mathbb{N}_0 \right\} \geq 1$$

1231 and  $s_m^+(\langle x \rangle^+, \langle y \rangle^+) \geq 1$ .

1232 *Statement* (c): To prove the lower bound, note that

$$|y^{-n}x^m| \geq |\pi_i(y^{-n}x^m)| = |-ny + mx|$$

1233 for all  $n, m \in \mathbb{N}_0$ . This implies  $s_m^+(\langle x \rangle^+, \langle y \rangle^+) \geq s_a^+(\langle x \rangle^+, \langle y \rangle^+)$ .

1234 Now we prove the upper bound. Set  $\alpha = s_a^+(\langle x \rangle^+, \langle y \rangle^+)$ . Without loss of gener-  
1235 ality we may assume that  $\alpha < 1$ . Furthermore, we may scale  $x$  and  $y$  by positive  
1236 constants without changing the value of  $s_a^+(\langle x \rangle^+, \langle y \rangle^+)$  or of  $s_m^+(\langle x \rangle^+, \langle y \rangle^+)$ . Hence  
1237 we may assume that  $\|y\| = 1$  and  $y$  is orthogonal to  $x - y$  with respect to the  
inner product on  $V_i$  associated with  $\|\cdot\|$ , see Figure 3. As a consequence we get

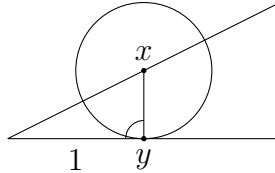


FIGURE 3. The constraints for the choice of  $x$  and  $y$ .

1238  $1 = \|y\| \leq \|x\|$  and  $\|x - y\| = \alpha^i \|x\|$  (due to Lemma B.8). Then  $1 = |y| \leq |x|$  and  
1239  $|x - y| = \alpha|x|$ . By Lemma B.10 we get

$$|y^{-n}x^n| \leq MQ\alpha^{i/c}|x^n|$$

for all  $n \geq 0$ . Thus

$$s_m^+(\langle x \rangle^+, \langle y \rangle^+) \leq \limsup_{n \rightarrow \infty} \frac{|y^{-n}x^n|}{|x^n|} \leq MQ\alpha^{i/c}. \quad \square$$

## REFERENCES

- [Ale02] Georgios K. Alexopoulos, *Random walks on discrete groups of polynomial volume growth*, Ann. Probab. **30** (2002), no. 2, 723–801, doi:10.1214/aop/1023481007, MR1905856 (2003d:60010), Zbl1023.60007. 26
- [Bab77] László Babai, *Some applications of graph contractions*, J. Graph Theory **1** (1977), no. 2, 125–130, Special issue dedicated to Paul Turán, doi:10.1002/jgt.3190010207, MR0460171 (57 #167), Zbl0381.05029. 20
- [BH99] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999, MR1744486 (2000k:53038), Zbl0988.53001. 2, 10, 28
- [Bou66] Nicolas Bourbaki, *Elements of mathematics. General topology. Part 1*, Hermann, Paris, 1966, MR0205210 (34 #5044a), Zbl0301.54001. 29
- [Bre12] Emmanuel Breuillard, *Geometry of locally compact groups of polynomial growth and shape of large balls*, preprint, 2012. 33, 34
- [BRW07] C. Paul Bonnington, R. Bruce Richter, and Mark E. Watkins, *Between ends and fibers*, J. Graph Theory **54** (2007), no. 2, 125–153, doi:10.1002/jgt.20202, MR2285455 (2007k:05100), Zbl1118.05017. 2
- [CG90] Lawrence J. Corwin and Frederick P. Greenleaf, *Representations of nilpotent Lie groups and their applications. Part I*, Cambridge Studies in Advanced Mathematics, vol. 18, Cambridge University Press, Cambridge, 1990, Basic theory and examples, MR1070979 (92b:22007), Zbl0704.22007. 30, 32
- [dlH00] Pierre de la Harpe, *Topics in geometric group theory*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 2000, MR1786869 (2001i:20081), Zbl0965.20025. 27, 28
- [Fre42] Hans Freudenthal, *Neuaufbau der Endentheorie*, Ann. of Math. (2) **43** (1942), 261–279, doi:10.2307/1968869, MR0006504 (3,315a), Zbl0060.40006. 2
- [GIS<sup>+</sup>89] Chris D. Godsil, Wilfried Imrich, Norbert Seifter, Mark E. Watkins, and Wolfgang Woess, *A note on bounded automorphisms of infinite graphs*, Graphs Combin. **5** (1989), no. 4, 333–338, doi:10.1007/BF01788688, MR1032384 (91c:05092), Zbl0714.05029. 21, 22
- [Glu55] Viktor M. Gluškov, *Locally nilpotent locally bicomact groups*, Trudy Moskov. Mat. Obšč. **4** (1955), 291–332, MR0072422 (17,281b), Zbl0068.01901. 31
- [Goo76] Roe W. Goodman, *Nilpotent Lie groups: structure and applications to analysis*, Lecture Notes in Mathematics, Vol. 562, Springer-Verlag, Berlin, 1976, MR0442149 (56 #537), Zbl0347.22001. 30, 33
- [Gro93] Mikhael Gromov, *Asymptotic invariants of infinite groups*, Geometric group theory, Vol. 2 (Sussex, 1991), London Math. Soc. Lecture Note Ser., vol. 182, Cambridge Univ. Press, Cambridge, 1993, pp. 1–295, MR1253544 (95m:20041), Zbl0841.20039. 2, 28
- [Gui73] Yves Guivarc’h, *Croissance polynomiale et périodes des fonctions harmoniques*, Bull. Soc. Math. France **101** (1973), 333–379, MR0369608 (51 #5841), Zbl0294.43003, [http://www.numdam.org/item?id=BSMF\\_1973\\_\\_101\\_\\_333\\_0](http://www.numdam.org/item?id=BSMF_1973__101__333_0). 26, 33
- [Gui80] ———, *Sur la loi des grands nombres et le rayon spectral d’une marche aléatoire*, Conference on Random Walks (Kleeback, 1979) (French), Astérisque, vol. 74, Soc. Math. France, Paris, 1980, pp. 47–98, 3, MR588157 (82g:60016), Zbl0448.60007. 26, 27
- [Hal64] Rudolf Halin, *Über unendliche Wege in Graphen*, Math. Ann. **157** (1964), 125–137, doi:10.1007/BF01362670, MR0170340 (30 #578), Zbl0125.11701. 2
- [Hoc65] Gerhard P. Hochschild, *The structure of Lie groups*, Holden-Day Inc., San Francisco, 1965, MR0207883 (34 #7696), Zbl0131.02702. 27, 30

- [Hop44] Heinz Hopf, *Enden offener Räume und unendliche diskontinuierliche Gruppen*, Comment. Math. Helv. **16** (1944), 81–100, doi:10.1007/BF02568567, MR0010267 (5,272e), Zbl0060.40008. 2
- [HR79] Edwin Hewitt and Kenneth A. Ross, *Abstract harmonic analysis. Vol. I*, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 115, Springer-Verlag, Berlin, 1979, Structure of topological groups, integration theory, group representations, MR551496 (81k:43001), Zbl0416.43001. 12, 14, 27, 29
- [JN95] Heinz A. Jung and Peter Niemeyer, *Decomposing ends of locally finite graphs*, Math. Nachr. **174** (1995), 185–202, doi:10.1002/mana.19951740113, MR1349044 (96h:05120), Zbl0833.05052. 2
- [Jun93] Heinz A. Jung, *Notes on rays and automorphisms of locally finite graphs*, Graph structure theory (Seattle, WA, 1991), Contemp. Math., vol. 147, Amer. Math. Soc., Providence, RI, 1993, pp. 477–484, MR1224725 (94d:05075), Zbl0787.05048. 2
- [Kai91] Vadim A. Kaimanovich, *Poisson boundaries of random walks on discrete solvable groups*, Probability measures on groups, X (Oberwolfach, 1990), Plenum, New York, 1991, pp. 205–238, MR1178986 (94m:60014), Zbl0823.60006. 26, 27
- [KLS12] Bernhard Krön, Jörg Lehnert, and Maya J. Stein, *Linear boundary and HNN-extensions*, preprint, 2012. 12, 13
- [Mal51] Anatolii I. Mal'tsev, *On a class of homogeneous spaces*, Amer. Math. Soc. Translation **1951** (1951), no. 39, 33, MR0039734 (12,589e), ZblZbl 0034.01701. 16
- [MKS04] Wilhelm Magnus, Abraham Karrass, and Donald Solitar, *Combinatorial group theory*, second ed., Dover Publications Inc., Mineola, NY, 2004, Presentations of groups in terms of generators and relations, MR2109550 (2005h:20052), Zbl1130.20307. 32
- [Pan89] Pierre Pansu, *Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un*, Ann. of Math. (2) **129** (1989), no. 1, 1–60, doi:10.2307/1971484, MR979599 (90e:53058), Zbl0678.53042. 16
- [Sab64] Gert Sabidussi, *Vertex-transitive graphs*, Monatsh. Math. **68** (1964), 426–438, doi:10.1007/BF01304186, MR0175815 (31 #91), Zbl0136.44608. 21, 22
- [Seg83] Daniel Segal, *Polycyclic groups*, Cambridge Tracts in Mathematics, vol. 82, Cambridge University Press, Cambridge, 1983, doi:10.1017/CBO9780511565953, MR713786 (85h:20003), Zbl0516.20001. 31
- [Sei91a] Norbert Seifter, *Groups acting on graphs with polynomial growth*, Discrete Math. **89** (1991), no. 3, 269–280, doi:10.1016/0012-365X(91)90120-Q, MR1112445 (92g:05099), Zbl0739.05041. 20
- [Sei91b] ———, *Properties of graphs with polynomial growth*, J. Combin. Theory Ser. B **52** (1991), no. 2, 222–235, doi:10.1016/0095-8956(91)90064-Q, MR1110471 (92i:05106), Zbl0668.05034. 20, 21
- [Str06] Markus Stroppel, *Locally compact groups*, EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zürich, 2006, doi:10.4171/016, MR2226087 (2007d:22001), Zbl1102.22005. 27
- [Tro83] Vladimir I. Trofimov, *Bounded automorphisms of graphs and the characterization of grids*, Izv. Akad. Nauk SSSR Ser. Mat. **47** (1983), no. 2, 407–420, MR697303 (84h:05067), Zbl0519.05037. 22
- [Tro84] ———, *Graphs with polynomial growth*, Mat. Sb. (N.S.) **123(165)** (1984), no. 3, 407–421, doi:10.1070/SM1985v051n02ABEH002866, MR735714 (85m:05041), Zbl0548.05033. 3, 20
- [Wil04] Stephen Willard, *General topology*, Dover Publications Inc., Mineola, NY, 2004, Reprint of the 1970 original [Addison-Wesley, Reading, MA; MR0264581], MR2048350, Zbl1052.54001. 23

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